Premium liability risks : modeling small claims

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Premium Liability Risks: Modeling Small Claims

1 Introduction

This note is motivated by the recent developments in risk management and risk-adjusted solvency requirements. We define a model for measuring premium liability risks for small claims in a non-life insurance company, which is closely related to the Swiss Solvency Test SST [5]. Our approach is neither new nor complicated, in fact there is a large variety of similar models using all kinds of different notations (see e.g. [2, 1, 4, 5]). Our goal is to give a mathematically rigorous framework to this problem and we introduce an understandable notation which has easy interpretations. This only allows for a fruitful discussion between practitioners and actuaries when setting the different parameters.

It is standard to split the total risk into different risk classes (see e.g. [4]): i) insurance risk, ii) market risk, iii) credit risk, and iv) operational risk. Each of these risk classes is then studied individually. The overall picture is obtained by applying an aggregation mechanism (which usually is complicated to capture dependencies between the different risk classes).

In non-life insurance we split insurance risks into further subclasses: premium liability risk (underwriting risk) and the claims provision risk (run-off risk). It has been observed, that it is useful to split premium liability risks into two subclasses (see e.g. SST [5]): a) large events, b) small claims.

a) Large events contain single large claims which are bigger than some limit (e.g. 5 Mio. CHF) as well as cumulative events. These are typically natural catastrophes like earthquake, windstorms, hailstorms, as well as man-made catastrophes like nuclear power accident, etc.

b) Each **small claim** is caused by an individual event, moreover we assume that small claims are smaller than an upper limit (e.g. 5 Mio. CHF), i.e. small claims exclude claims where a single event causes lots of small claims.

From experience, it has figured out that this split is useful, because the tail behaviour of the aggregated small claims is rather different from the tail behaviour of large events. Therefore it is difficult to model small claims and large events simultaneously, since the risk drivers have quite different sources.

In the present work we concentrate on modeling premium liability risks for small claims. The basic idea is to clearly distinguish between process risk and parameter risk which leads to a risk-adjusted model. Similar approaches can be found e.g. in [2, 4, 1, 5].

2 Premium Liability Model for Small Claims

2.1 *Single Line of Business*

We start with the description of a single line of business $i \in \{1, \ldots, I\}$. The total claims amount for small claims is denoted by $X^{(i)}$. The number of small claims in line i is denoted by $N^{(i)}$, and $V^{(i)}$ is an appropriate deterministic volume measure for line i (e.g. number of risks, etc.).
Furthermore we denote by $\mathbf{\Theta}^{(\mathrm{i})} = (\Theta_1^{(i)}, \Theta_2^{(i)})$ the underlying risk characteristics of the accident year under consideration.

Model Assumption 2.1 (SST Model [5])

(A1) Conditionally, given $\mathbf{\Theta}^{(\mathrm{i})}$, $X^{(i)}$ is compound Poisson distributed, i.e.

$$X^{(i)} = \sum_{k=1}^{N^{(i)}} Y_k^{(i)}, \tag{2.1}$$

with

i) $N^{(i)}$ is Poisson distributed with mean $V^{(i)} \cdot \lambda_{(i)}(\Theta_1^{(i)})$, and

ii) the claims severities $Y_k^{(i)}$ are bounded, i.i.d. with

$$E\left[Y_k^{(i)} \middle| \mathbf{\Theta}^{(\mathrm{i})} \right] = \mu_{(i)}(\Theta_2^{(i)}), \tag{2.2}$$

$$\mathrm{Var}\left(Y_k^{(i)} \middle| \mathbf{\Theta}^{(\mathrm{i})} \right) = \sigma_{(i)}^2(\Theta_2^{(i)}), \tag{2.3}$$

where $\lambda_{(i)}(\cdot)$, $\mu_{(i)}(\cdot)$, $\sigma_{(i)}^2(\cdot)$ are positive functions of the random variables $\mathbf{\Theta}_i$.

(A2) $\Theta_1^{(i)}$ and $\Theta_2^{(i)}$ are independent random variables. □

Hence, we consider a mixed Poisson process, where also the claims severities depend on a latent variable $\Theta_2^{(i)}$.
We define the expected frequency and the expected number of claims in line i

$$\lambda_{(i)}^{(0)} = E\left[\lambda_{(i)}\left(\Theta_1^{(i)}\right)\right] \quad \text{and} \quad n_{(i)}^{(0)} = E\left[N^{(i)}\right] = V^{(i)} \cdot \lambda_{(i)}^{(0)} . \tag{2.4}$$

Hence the total expected claim amount for small claims in line i is given by

$$\begin{aligned} E\left[X^{(i)}\right] &= E\left[E\left[X^{(i)} \middle| \Theta^{(i)}\right]\right] \\ &= E\left[V^{(i)} \lambda_{(i)}\left(\Theta_1^{(i)}\right) \mu_{(i)}\left(\Theta_2^{(i)}\right)\right] \\ &= n_{(i)}^{(0)} \cdot E\left[Y_1^{(i)}\right] . \end{aligned}$$

Definition 2.2 *Assume X is a random variable with finite second moment. The coefficient of variation of X is defined as follows*

$$\text{Vco}(X) = \frac{\text{Var}^{1/2}(X)}{E[X]} . \tag{2.5}$$

Proposition 2.3 *Under Model Assumption 2.1 we have*

$$\text{Vco}^2\left(X^{(i)}\right) = (R_{Param}^{(i)})^2 + \frac{1}{n_{(i)}^{(0)}} \left(\text{Vco}^2\left(Y_1^{(i)}\right) + 1\right) , \tag{2.6}$$

where

$$\begin{aligned} R_{Param}^{(i)} = \Big(& \text{Vco}^2\left(\lambda_{(i)}\left(\Theta_1^{(i)}\right)\right) + \text{Vco}^2\left(\mu_{(i)}\left(\Theta_2^{(i)}\right)\right) \\ & + \text{Vco}^2\left(\lambda_{(i)}\left(\Theta_1^{(i)}\right)\right) \cdot \text{Vco}^2\left(\mu_{(i)}\left(\Theta_2^{(i)}\right)\right) \Big)^{1/2} . \end{aligned} \tag{2.7}$$

Remarks:

- **Parameter Risk.** $R_{Param}^{(i)}$ denotes risks in the parameter estimates. It is a measure for: 'how good' can an actuarial estimate at most be? At the beginning of the year the actuary has to estimate the future frequency and the future claims averages per line of business i ('best estimate'). There are several external factors which make his life difficult, because they do not diversify (no matter how large the insurance portfolio is), e.g. he has to predict inflation, weather conditions, etc. All these external factors are gathered by the risk characteristics $\Theta^{(i)}$, which tells us how the frequency λ and the claims average μ vary from year to year. Typically in non-life insurance $R^{(i)}_{Param}$ is within 3% to 6%. (The Swiss Solvency Test [5] gives default values for different lines of business. These default values were estimated with the help of market insurance data.) In fact, formula (2.7) even breaks down the parameter risk into the uncertainty coming from the estimates of the frequencies and the claims averages, respectively.

- **Process Risk.** The second part on the right-hand side of (2.6) denotes the process risk. It measures, how large the fluctuation within the portfolio is. Of course, this is the diversifiable part, the larger the portfolio $V^{(i)}$, the larger the expected number of claims $n^{(0)}_{(i)}$, the smaller the process risk.

  $\mathrm{Vco}(Y_1^{(i)})$ is the coefficient of variation of a single claim. It has turned out that this is a rather stable number across different insurance companies. Typically, if we only take claims smaller than 5 Mio. CHF, it is within 2 (health business) and 11 (general liablity). Claims bigger than 5 Mio. CHF are considered in the risk class 'large claims/events' (see [5]).

- Provided the parameter risk $R^{(i)}_{Param}$ and the coefficient of variation of a single claim $Y_1^{(i)}$, we can easily estimate the coefficient of variation and the standard deviation of the total claim amount of small claims $X^{(i)}$ per line of business i. The coefficient of variation of $X^{(i)}$ is decreasing in $n^{(0)}_{(i)}$ and bounded below by $R^{(i)}_{Param}$, which is the part that does not diversify (see also Figure 1 below).

Proof of Proposition 2.3. Using the standard approach

$$\mathrm{Var}\left(X^{(i)}\right) = \mathrm{Var}\left(E\left[X^{(i)}\middle|\Theta^{(i)}\right]\right) + E\left[\mathrm{Var}\left(X^{(i)}\middle|\Theta^{(i)}\right)\right], \qquad (2.8)$$

we obtain the decomposition into parameter risk and process error. An exercise with conditional expectation then easily leads to

$$\operatorname{Var}\left(E\left[X^{(i)}\middle|\Theta^{(i)}\right]\right) = \left(R^{(i)}_{Param}\right)^2 \cdot E\left[X^{(i)}\right]^2, \tag{2.9}$$

$$E\left[\operatorname{Var}\left(X^{(i)}\middle|\Theta^{(i)}\right)\right] = \frac{E\left[X^{(i)}\right]^2}{n^{(0)}_{(i)}}\left(\operatorname{Vco}^2\left(Y_1^{(i)}\right)+1\right). \tag{2.10}$$

This finishes the proof. □

2.2 *Portfolio with I Lines of Business*

Now we define the model which allows to aggregate single lines of business $i \in \{1,\dots,I\}$. As in [1], we assume that the correlation between the different lines of business is only applicable on the parameter risk part. Here we slightly differ from the SST approach (see [5, 3]).

Model Assumption 2.4 (Premium Liability Model for Small Claims)

(B1) Conditionally, given $\Theta = \left(\Theta^{(1)},\dots,\Theta^{(I)}\right)$, $\{X^{(1)},\dots,X^{(I)}\}$ are independent random variables which are compound Poisson distributed satisfying (A1) of Model Assumptions 2.1 for all $i \in \{1,\dots,I\}$.

(B2) The random variable $\Theta_1^{(i)}$ and $\Theta_2^{(j)}$ are independent for all $i, j \in \{1,\dots,I\}$.

□

Assume X, Y are two r.v. with finite variance. The correlation of X and Y satisfies

$$\begin{aligned}\rho(X,Y) = \operatorname{Corr}(X,Y) &= \frac{\operatorname{Cov}(X,Y)}{\operatorname{Var}(X)^{1/2}\cdot\operatorname{Var}(Y)^{1/2}} \\ &= \frac{\operatorname{Cov}(X,Y)}{E[X]\cdot E[Y]}\cdot\operatorname{Vco}^{-1}(X)\cdot\operatorname{Vco}^{-1}(Y).\end{aligned} \tag{2.11}$$

Proposition 2.5 *Under Model Assumption 2.4 we have for* $i \neq j$

$$\frac{\operatorname{Cov}\left(X^{(i)},X^{(j)}\right)}{E\left[X^{(i)}\right]\cdot E\left[X^{(j)}\right]} = \rho^{\lambda}_{ij} + \rho^{\mu}_{ij} + \rho^{\lambda}_{ij}\cdot\rho^{\mu}_{ij}, \tag{2.12}$$

where

$$\rho_{ij}^{\lambda} = \rho\left(\lambda_{(i)}(\Theta_1^{(i)}), \lambda_{(j)}(\Theta_1^{(j)})\right) \cdot \mathrm{Vco}\left(\lambda_{(i)}(\Theta_1^{(i)})\right) \cdot \mathrm{Vco}\left(\lambda_{(j)}(\Theta_1^{(j)})\right), \tag{2.13}$$

$$\rho_{ij}^{\mu} = \rho\left(\mu_{(i)}(\Theta_2^{(i)}), \mu_{(j)}(\Theta_2^{(j)})\right) \cdot \mathrm{Vco}\left(\mu_{(i)}(\Theta_2^{(i)})\right) \cdot \mathrm{Vco}\left(\mu_{(j)}(\Theta_2^{(j)})\right). \tag{2.14}$$

Remarks:

- We see that we only assume correlation between the parameter errors (as in [1]). This is a slight difference from the standard SST model [5] which uses a shortcut to estimate the overall variance. On the one hand, this shortcut can easily be implemented, but on the other hand it underestimates the diversification effect for small portfolios between the lines of business.
- In order to determine the second moment of the total premium liability distribution for small claims we need to estimate the uncertainties in the parameter estimates of λ and μ and the correlations between these estimates. Unfortunately, in the SST there is no explicit default value given for the parameters $\rho(\lambda_{(i)}(\Theta_1^{(i)}), \lambda_{(j)}(\Theta_1^{(j)}))$, $\rho(\mu_{(i)}(\Theta_2^{(i)}), \mu_{(j)}(\Theta_2^{(j)}))$, $\mathrm{Vco}\left(\lambda_{(i)}(\Theta_1^{(i)})\right)$ and $\mathrm{Vco}\left(\mu_{(i)}(\Theta_2^{(i)})\right)$, but only for the overall parameter risk $R_{Param}^{(i)}$ (see (2.7)). Moreover, the SST provides a correlation matrix between lines of business, but it is not further specified what is exactly meant by the matrix.

  In general one should say, that the estimate of an appropriate correlation matrix is very difficult. Often data is not really of help, but one should rather set the correlation matrix with external (actuarial) know-how.

Proof of Proposition 2.5. Choose $i \neq j$. Using the standard approach

$$\begin{aligned}\mathrm{Cov}\left(X^{(i)}, X^{(j)}\right) &= \mathrm{Cov}\left(E\left[X^{(i)}\middle|\Theta\right], E\left[X^{(j)}\middle|\Theta\right]\right) \\ &\quad + E\left[\mathrm{Cov}\left(X^{(i)}, X^{(j)}\middle|\Theta\right)\right].\end{aligned}$$

The last term is 0 since we assume conditional independence of the compound Poisson distributed random variables $X^{(i)}$. Hence there remains to calculate the covariance of the conditional expectations. This is again an excercise using conditional expectations. □

Corollary 2.6 *Under Model Assumption 2.4 we have*

$$E\left(\sum_{i=1}^{I} X^{(i)}\right) = \sum_{i=1}^{I} n_{(i)}^{(0)}\ E[Y_1^{(i)}], \tag{2.15}$$

$$\begin{aligned} \mathrm{Var}\left(\sum_{i=1}^{I} X^{(i)}\right) &= \sum_{i,j=1}^{I} n_{(i)}^{(0)}\ n_{(j)}^{(0)}\ E[Y_1^{(i)}]\ E[Y_1^{(j)}]\ \left(\rho_{ij}^{\lambda} + \rho_{ij}^{\mu} + \rho_{ij}^{\lambda} \cdot \rho_{ij}^{\mu}\right) \\ &+ \sum_{i=1}^{I} n_{(i)}^{(0)}\ E[Y_1^{(i)}]^2\ \left(\mathrm{Vco}^2\left(Y_1^{(i)}\right) + 1\right). \end{aligned} \tag{2.16}$$

Remarks:

- The first term on the right-hand side of (2.16) grows quadratic in the volume $n_{(i)}^{(0)}$ (non-diversifiable part), whereas the second grows linear in the volume.
- We have determined the first two moments of the premium liability distribution for small claims. To obtain the whole distribution, one would have to specify the distribution of the risk characteristics Θ and the conditional distributions of the claims severities. In order to avoid these difficulties and moreover to avoid complex discussions about dependence structures and copulas within the random vector Θ, we simply assume that $\sum_i X^{(i)}$ is lognormally distributed with the first two moments given by Corollary 2.6. In general, it is difficult to statistically justify this choice, but this is a very common approach in practice (see e.g. [1, 5]).

3 Example

We choose a portfolio with two lines of business: 1) motor third party liability (MTPL), 2) motor hull (M Hull). We assume

Line of business i	$Vco\left(\lambda_{(i)}(\Theta_1^{(i)})\right)$	$\mathrm{Vco}\left(\mu_{(i)}(\Theta_2^{(i)})\right)$	$R_{Param}^{(i)}$	$E[Y_1^{(i)}]$	$\mathrm{Vco}(Y_1^{(i)})$
MTPL, $i=1$	2.50%	3.00%	3.91%	8'000	9.0
M Hull, $i=2$	3.00%	2.00%	3.61%	3'000	3.0

In Figure 1 we see that the coefficient of variation of $X^{(i)}$ decreases in the volume. Asymptotically it is equal to the parameter risk $R^{(i)}_{Param}$.

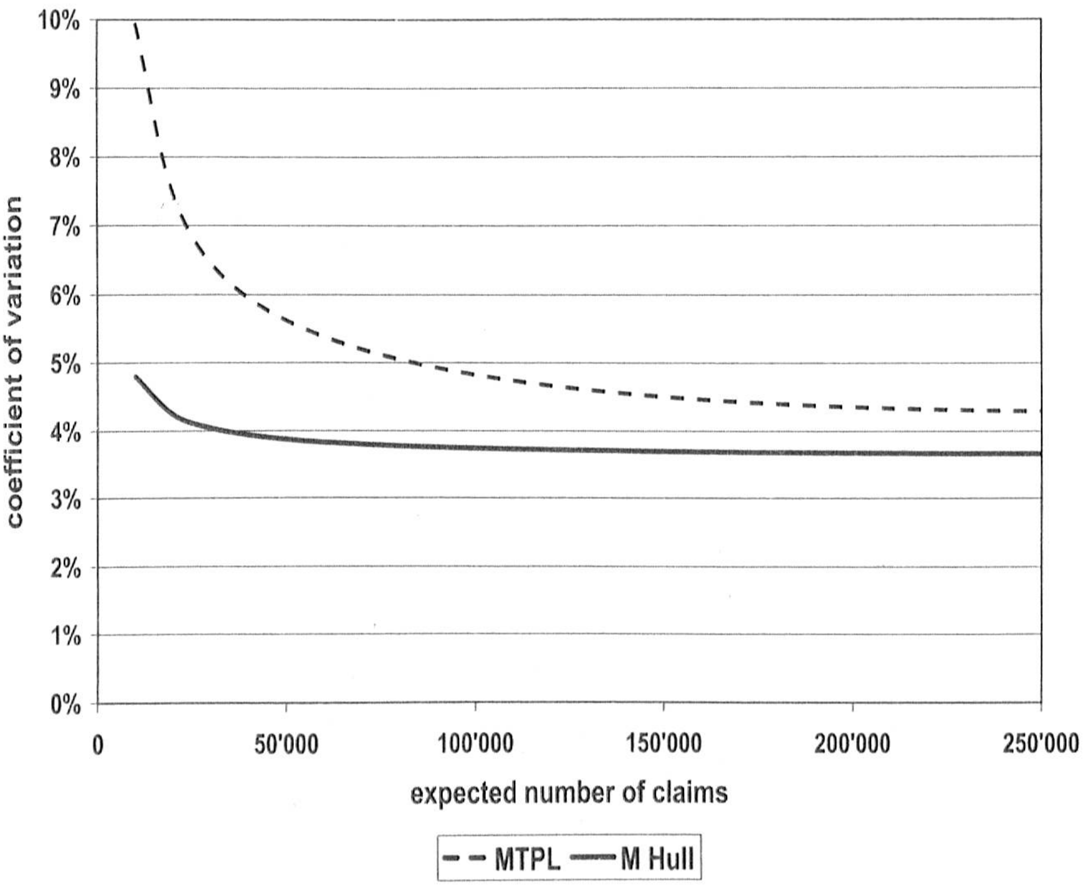

Figure 1: Stand-alone coefficients of variation $\mathrm{Vco}(X^{(i)})$ for different expected number of claims.

Now, we merge the two lines of business. Therefore we have to specify the correlation between the parameter estimates. We choose $\rho\big(\lambda_{(1)}(\Theta_1^{(1)}), \lambda_{(2)}(\Theta_1^{(2)})\big) = \rho\big(\mu_{(1)}(\Theta_2^{(1)}), \mu_{(2)}(\Theta_2^{(2)})\big) = 25\%$. Hence we have $\rho^{\lambda}_{12} = (1.37\%)^2$ and $\rho^{\mu}_{12} = (1.22\%)^2$. Next we define the matrix

$$R = (r_{ij})_{i,j=1,2} = \begin{pmatrix} \left(R^{(1)}_{Param}\right)^2 & \rho^{\lambda}_{12} + \rho^{\mu}_{12} + \rho^{\lambda}_{12} \cdot \rho^{\mu}_{12} \\ \rho^{\lambda}_{12} + \rho^{\mu}_{12} + \rho^{\lambda}_{12} \cdot \rho^{\mu}_{12} & \left(R^{(2)}_{Param}\right)^2 \end{pmatrix}$$

$$= \begin{pmatrix} (3.91\%)^2 & (1.84\%)^2 \\ (1.84\%)^2 & (3.61\%)^2 \end{pmatrix}. \tag{3.1}$$

Hence we have the following formula, which can easily be implemented on a spread-sheet:

$$\mathrm{Var}\left(X^{(1)}+X^{(2)}\right)=\sum_{i,j=1}^{2} n_{(i)}^{(0)} n_{(j)}^{(0)} E[Y_1^{(i)}]E[Y_1^{(j)}]r_{ij} + \sum_{i=1}^{2} n_{(i)}^{(0)} E[Y_1^{(i)}]^2 \left(\mathrm{Vco}^2\left(Y_1^{(i)}\right)+1\right). \quad (3.2)$$

Question: Which is the optimal portfolio-mix (minimal variance) for a fixed volume? We could either fix the total expected number of claims $n_{(1)}^{(0)}+n_{(2)}^{(0)}=n$ or the total expected claims amount for small claims $n_{(1)}^{(0)}E[Y_1^{(1)}]+n_{(2)}^{(0)}E[Y_1^{(2)}]=M$. The analysis is essentially the same in both cases, but the results differ since the weighted volumes are different. We choose the second alternative, optimize the portfolio-mix for a fixed expected claims amount M:
Choose $\alpha \in [0,1]$ such that

$$n_{(1)}^{(0)}E[Y_1^{(1)}]=\alpha M \quad \text{and} \quad n_{(2)}^{(0)}E[Y_1^{(2)}]=(1-\alpha)M. \quad (3.3)$$

Hence we have

$$\begin{aligned} f_M(\alpha) &= \mathrm{Var}\left(X^{(1)}+X^{(2)}\right) \\ &= M^2\left(\alpha^2 r_{11}+(1-\alpha)^2 r_{22}+2\alpha(1-\alpha)r_{12}\right) \\ &\quad +\alpha M E[Y_1^{(1)}]\left(\mathrm{Vco}^2(Y_1^{(1)})+1\right) \\ &\quad +(1-\alpha)M E[Y_1^{(2)}]\left(\mathrm{Vco}^2(Y_1^{(2)})+1\right). \end{aligned} \quad (3.4)$$

The question is now, which is the optimal $\alpha \in [0,1]$ which minimizes $f_M(\alpha)$? If α which minimizes $f_M(\alpha)$ does not lie within $[0,1]$, then we prefer to have only one line of business.
The optimal portfolio-mix for $M=600$ Mio. is given at (see Figure 2)

$$\begin{aligned} \alpha_M &= \frac{r_{22}-r_{12}}{r_{11}+r_{22}-2r_{12}} \\ &\quad + \frac{E[Y_1^{(2)}]\left(\mathrm{Vco}^2(Y_1^{(2)})+1\right)-E[Y_1^{(1)}]\left(\mathrm{Vco}^2(Y_1^{(1)})+1\right)}{2M(r_{11}+r_{22}-2r_{12})} \\ &= 20.5\%, \end{aligned} \quad (3.5)$$

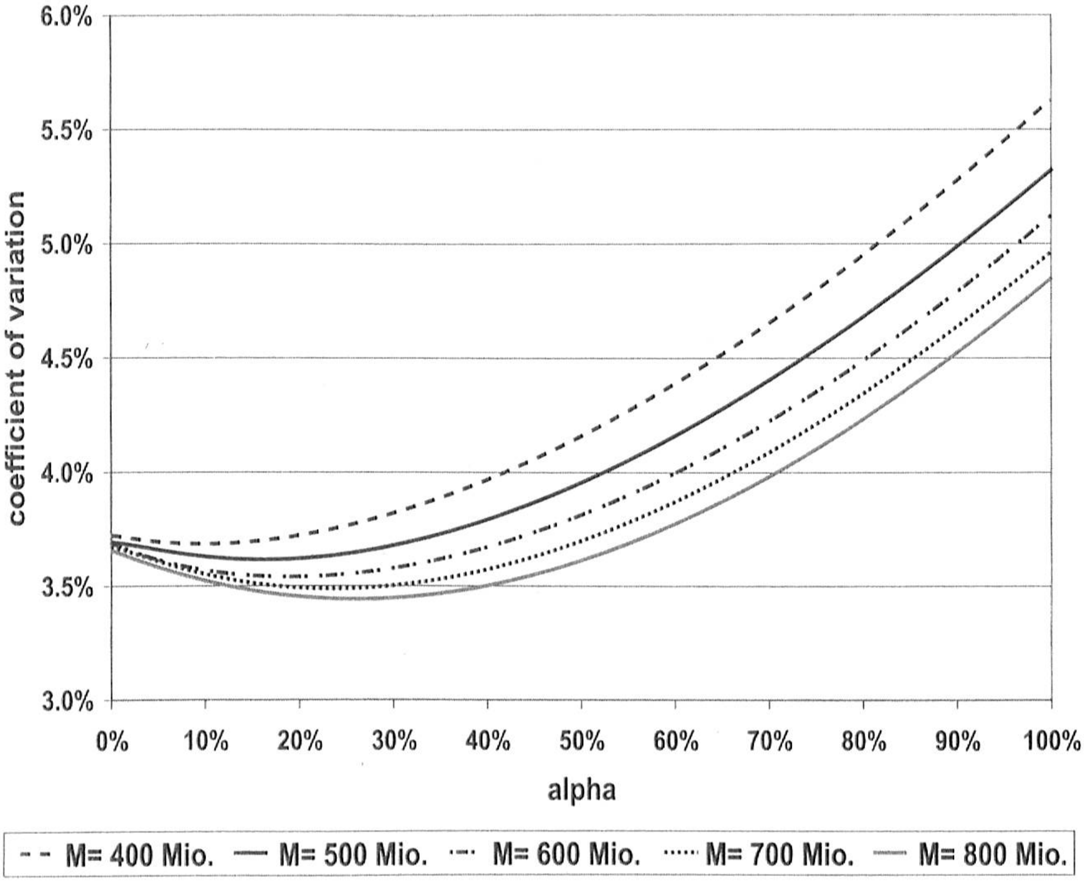

Figure 2: $\mathrm{Vco}(X^{(1)} + X^{(2)})$ for different total expected claims amounts M.

which proposes that MTPL is 20.5% of the total volume and M Hull is 79.5% of the total volume in order to minimize the coefficient of variation.

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Abstract

This work focuses on modeling premium liability risks for small claims in a non-life insurance company (see e.g. Swiss Solvency Test [5]). We give a mathematically rigorous framework and introduce notations which have easy interpretations. The theory is then applied to an example where we optimize a portfolio with two lines of business.

Zusammenfassung

Diese Arbeit behandelt die stochastische Modellierung des Normalschadenaufwandes für Geschäftsjahresfälle in einer Nicht-Leben Versicherungsgesellschaft (vgl. auch Swiss Solvency Test [5]). Wir formulieren ein mathematisches Modell mit leicht verständlichen Notationen. Am Ende der Arbeit wenden wir unsere Theorie auf ein Portfolio-Optimierungsproblem an (für ein Portfolio, welches aus zwei Branchen besteht).

Résumé

Le présent travail traite de la modélisation du montant des senistres légers de l'année courante dans une companie d'assurances non-vie (voir également Swiss Solvency Test [5]). Nous formulons un modèle mathématiquement rigoureux et introduisons des notations à l'interprétation aisée. La théorie est ensuite appliquée à un example dans lequel nous optimisons un portfeuille comportant deux secteurs d'activité.