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A new approach to the "Cosmological Constant" problem

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Abstract. We present a generalization of the "cosmological no-hair theorem" to a wide class of nonminimally coupled scalar-tensor theories of gravity through which it is possible to define a time–dependent "cosmological constant".

The so called "*cosmological no–hair theorem*" [1], can be generalized to a wide class of nonminimally coupled scalar–tensor theories of gravity (see [2] and references therein). In this way we get also the possibility of introducing a time–dependent "cosmological constant" [3]. The no–hair conjecture was introduced firstly by Hoyle and Narlikar [4]. In 1983, Wald gave a proof of a simplified version of that conjecture. Precisely he proved that: *All Bianchi cosmologies (except IX), in presence of a positive cosmological constant, asymptotically approach the de Sitter behaviour* [1]. In all these discussions (in Wald's paper too), the "cosmological constant" is a true constant and it is put by hand in the gravitational arena. Here we discuss how to introduce an "effective cosmological constant" in the context of nonminimally coupled scalar–tensor theories of gravity, being also present a standard perfect fluid matter (non–interacting with the scalar field ϕ). In this way the "cosmological constant" turns out to be time–dependent, becoming only asymptotically a true constant. Therefore we focuse our attention on the following question: how is it possible to construct a time–dependent "cosmological constant" coherently with the Einstein equations as well as with the (contracted) Bianchi identity? In other words, for introducing an effective (time–dependent) "cosmological constant", we cannot refer to the standard stress–energy tensor in the form $\Lambda g_{\mu\nu}$ since this implies the introduction of a true constant Λ. We will consider here only FRW-flat cosmologies. In all these theories, it is introduced an "effective gravitational constant", which is, in our units ($8\pi G_N = \hbar = c = 1$), $G_{eff} = -1/2F(\phi)$, where $F(\phi)$ describes the (nonminimal) gravitational coupling. The Einstein equations are

$$H^2 + \frac{\dot{F}}{F}H + \frac{\rho_\phi}{6F} + \frac{\rho_m}{6F} = 0\,, \tag{1}$$

$$\dot{H} = -\left(H^2 + \frac{V}{6F}\right) - H\frac{\dot{F}}{2F} + \frac{\dot{\phi}^2}{6F} - -\frac{1}{2}\frac{\ddot{F}}{F} + \frac{3p_m + \rho_m}{12F}. \tag{2}$$

where $H = \dot{a}/a$, $\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)$, ρ_m, p_m are, respectively, the Hubble parameter, the energy density of scalar field (the potential is generic), the energy density and the pressure of standard matter. Eq. (1) can be rewritten using the second degree polinomial $\mathcal{P}(H)$, as: $\mathcal{P}(H) \equiv (H - \Lambda_{eff,1})(H - \Lambda_{eff,2}) = -\rho_m/6F$, being

$$\Lambda_{eff,1,2} = -\frac{\dot{F}}{2F} \pm \sqrt{\left(\frac{\dot{F}}{2F}\right)^2 - \frac{\rho_\phi}{6F}}\;; \tag{3}$$

where "1" is relative to the plus sign and "2" to the minus. The two $\Lambda_{eff,1,2}$ have to be real, then the restriction $(\dot{F}/2F)^2 \geq \rho_\phi/6F$ has to be satisfied. It is immediate to see that $\Lambda_{eff,1} + \Lambda_{eff,2} = -\dot{F}/F$ and that $\Lambda_{eff,1} - \Lambda_{eff,2} = 2\sqrt{\left(\dot{F}/2F\right)^2 - \rho_\phi/6F} \geq 0$. That is, in general: $\Lambda_{eff,1} \geq \Lambda_{eff,2}$. Of course the equation for $\dot{H}$ can be also written using $\mathcal{P}(H)$. We make now the following hypothesis, i.e. for $t \gg 0$, we suppose that: **i)** $\dot{F}/F \longrightarrow \Sigma_0$, and **ii)** $\rho_\phi/6F(\phi) \longrightarrow \Sigma_1$, where $\Sigma_{0,1}$ are two constants depending on the parameters of the coupling and the potential. Under these two hypotheses we see that the two quantities $\Lambda_{eff,1,2}$ are asymptotically constants. *Viceversa*, if we assume that $\Lambda_{eff,i} \rightarrow \Lambda_i$ (constants), we see that $\dot{F}/F$ and $\rho_\phi/6F$ become constants for large t. Then hypotheses **i)** and **ii)** are necessary and sufficient conditions since the two Λ's are asymptotically constants. We will also assume, that asymptotically the sign of $F(\phi)$ is constant (this is our third, quite natural, assumption). Here we consider the (physical) case: $F(t \gg 0) \leq 0$. Since we are considering that, asymptotically, $\dot{F}/F$ is constant, the above case has two subcases related to the sign of $\dot{F}$. Let us consider the case $\dot{F}(t \gg 0) \leq 0$; from hypothesis **i)** we have $\Sigma_0 \geq 0$. Furthermore the reality condition on the discriminant connected to the Λ_{eff}'s is (asymptotically) satisfied. For $\mathcal{P}(H)$ we get the disequality $\mathcal{P}(H) \geq 0$, then we have $H \geq \Lambda_1$, $H \leq \Lambda_2$. For the Λ_i, we obtain the asymptotic expressions: $\Lambda_{1,2} = -\Sigma_0/2 \pm \sqrt{(\Sigma_0/2)^2 + |\Sigma_1|} \geq 0$. For $\dot{H}$, we get:

$$\dot{H} = -\left(H^2 - \frac{V}{6|F|}\right) - H\frac{\dot{F}}{2F} - \frac{\dot{\phi}^2}{6|F|} - \frac{1}{2}\left(\frac{\ddot{F}}{F}\right) - \frac{3p_m + \rho_m}{12|F|}\,. \tag{4}$$

If (this is our last hypothesis) **iii)** $H^2 \geq V/6|F|$ we obtain then $\dot{H} \leq 0$. In other words $H(t)$ has a horizontal asymptote, or, equivalently, H goes to a constant. This can be seen in the following way: the conditions found on H and $\dot{H}$ imply that we can construct the sequence $\sigma_n = H(t_0 + t_n)$, where t_0 is an arbitrarily chosen large time and t_n is an ordered increasing sequence of instants greater than t_0 for each n. Then we can say that the sequence σ_n is such that

$$\sigma_{n+1} \leq \sigma_n \quad \text{for any } n \geq 0, \quad (\text{being } \dot{H} \geq 0) \tag{5}$$

and

$$\sigma_n \geq \Lambda_{eff,1}(t_n > t_0) \text{ for any } n \geq 0. \tag{6}$$

That is, σ_n is a monotone decreasing sequence, then it is convergent; that is $\lim_{t\to\infty} H(t) = H_0$, thus H has a horizontal asymptote.

Then the universe, for large t, has a de Sitter behaviour, (i.e. $a(t) \sim \exp(\alpha t)$, where α is an unknown constant). The universe, for large t, expands as de Sitter, although our conditions do not fix the parameter which specifies such an expansion, i.e. the effective "cosmological constant". If we compare Wald's conditions with ours, we have (of course asymptotically):

$$\begin{array}{ccc} \text{(Wald)} & & \text{(our conditions)} \\ \left(H-\sqrt{\frac{\Lambda}{3}}\right)\left(H+\sqrt{\frac{\Lambda}{3}}\right) \geq 0 & \Longleftrightarrow & (H-\Lambda_1)(H-\Lambda_2) \geq 0, \\ \dot{H} \leq \frac{\Lambda}{3} - H^2 \leq 0 & \Longrightarrow & \dot{H} \leq 0. \end{array}$$

The equations involving H are the same in both cases. The true difference concerns the equation for $\dot{H}$; our condition ($\dot{H} \leq 0$) is more general than $\dot{H} \leq (\Lambda/3 - H^2) \leq 0$. The hypothesis **iii)**, when $\phi \to$ *const.*, is nothing else but $H^2 \geq \Lambda/3$, that is in this case we recover the standard case $V = const.$ By some algebra, it is easy to show that such a hypothesis is equivalent to $\dot{\phi}^2/24FV \geq (F'/F)^2 = \left(G'_{eff}/G_{eff}\right)^2$. Having shown that $a(t)$ behaves like de Sitter for large t, we have to see if it is possible to fix α in order to recover the effective "cosmological constant". To this purpose, the Bianchi contracted identity for matter is needed (it is important to stress that we have not used it to find the asymptotic behaviour of $a(t)$). As usual, we get $\rho_m = Da^{-3\gamma}$ (we have used the standard matter state equation; D is the integration constant giving the matter content of universe). Introducing this expression for the matter in Eq. (2), for large t, we have $(H - \Lambda_1)(H + |\Lambda_2|) = D|F_0|^{-1}e^{-(3\gamma\alpha+\Sigma_0)t}$, being $3\gamma\alpha + \Sigma_0 \geq 0$. Then we get $(H - \Lambda_1)(H + |\Lambda_2|) \to 0$ i.e. $H \to \Lambda_1$. The (effective) matter content, $\rho_m/6F(\phi)$, tells us how H is "distant" from the de Sitter behaviour given by the cosmological constant Λ_1. In other words, we do not use the Bianchi identity for finding the type of expansion, we use it only to select (asymptotically) the specific value of what we call "cosmological constant". Essentially, we have introduced the effective "cosmological constant" in a "pragmatic" way, through the (asymptotic) de Sitter behaviour of $a(t)$. In a certain sense, the approach in [1] is reversed: there, Λ (constant) is introduced *a-priori* and this leads, under certain hypotheses, to a de Sitter expansion. Here, the de Sitter expansion is recovered under different hypotheses (in all these models the energy condition theorems are not valid), which (together with the contracted Bianchi identity for matter) select the effective "cosmological constant". Moreover, we have obtained such a result without assuming to recover the standard gravity (i.e. we do not need that $G_{eff} \to G_N$). Considering also the Klein–Gordon equation:

$$\ddot{\phi} + 6\left(\dot{H} + 2H^2\right)F'(\phi) + 3H\dot{\phi} + V'(\phi) = 0\,, \tag{7}$$

we get, for large t, $\dot{\phi}^2/F(\phi) = const = C_1(\Sigma_0, \Sigma_1)$, which implies that $|\Sigma_1| \geq 2\Sigma_0^2$. We get also $V/6F = C_2(\Sigma_0, \Sigma_1)$, as well as $\dot{\phi}^2/V = C_3(\Sigma_0, \Sigma_1)$. Concerning the other case, $\dot{F}(\phi(t \gg 0)) \geq 0$, everything goes as in the Wald case.

As an examples we restrict our analysis to a dust–dominated universe since we are interested in asymptotic regimes. The case we discuss is the following: $F(\phi) = k_0\phi^2$, $V(\phi) = \lambda\phi^2$, $\gamma = 1$, where $k_0 < 0$ and $\lambda > 0$ are free parameters, the de Sitter regime is recovered even if we do not recover the standard gravity. The coupling $F(\phi)$ is always negative, whereas $V(\phi)$ is always positive and $\dot{F}(\phi(t \gg 0)) < 0$. In fact the general solution [2], [3] is de Sitter for $t \gg 0$ being $a(t) \sim \exp(\Lambda t)$ where $\Lambda = \sqrt{\lambda(1-8k_0)^2/[2k_0(12k_0-1)(3-32k_0)]} > 0$, is exactly the asymptotic behaviour of $\Lambda_{eff,1}$ (the scalar field also ϕ has an exponential behaviour). It is relevant to stress that $F(\phi(t))$ diverges. We do not recover asymptotically the standard G_N. Actually we have, at plus infinity, a sort of asymptotic gravitational freedom [5]: nevertheless we have a de Sitter behaviour for $a(t)$. Furthermore, the condition **ii)** is always satisfied.

In conclusion, it is possible to formulate a cosmic no–hair theorem in the framework of nonminimally coupled scalar–tensor theories by introducing a time–dependent "cosmological constant", not using the "geometrical side" of such theories (i.e. $\Lambda g_{\mu\nu}$, as usual) but their "scalar side". Under the hypotheses we used, the de Sitter asymptotic regime is obtained and this is not necessarily connected with recovering the standard gravity. It is interesting to stress that, by this mechanism, the "amount of Λ" is strictly related to the matter content of the universe. This could have interest in connection with the Ω problem. Finally, this way of introducing a time–dependent Λ could be useful to obtain a large value of the "cosmological constant" in the early epoch, necessary to have inflation, and a small value in the present epoch (see also [6]).

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