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Autor(en): **Amiet, J.-P. / Huguenin, P.**

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Currents and local currents in Galilean quantum mechanics

by

J.-P. Amiet and **P. Huguenin**

Institut de Physique de l'Université de Neuchâtel, Rue A.-L. Breguet 1, CH-2000 Neuchâtel (Switzerland)

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Abstract. We give a general expression for the probability current of a non-relativistic quantum system characterized by an arbitrary Hamiltonian. This current satisfies the continuity equation and is linear in both velocity and probability density. Locality, characterized by the vanishing of j in domains of vanishing probability, is discussed. These topics are also discussed for the phase space quantities quasi-current and quasi-probability.

1. Introduction

It is well known that self-adjoint generators give rise to a unitary time evolution of states. In these circumstances the total probability is conserved and there exists an infinity of different currents of probability that obey the continuity equation. In spite of this large manifold, it is not a trivial matter to give explicitely a single example in the general case. We first solve here this problem and give a general expression for the current that reduces to the usual formula in the case of the ordinary Schrödinger equation (Galilean particle in external electro-magnetic fields). In the classical limit $\hbar \to 0$ this current is exactly the classical one.

The calculation method is based on the Weyl–Wigner transcription [1] of quantum mechanics in terms of phase-space functions. The quasi-probability current in phase space is first defined. Then, the projection into configuration space by integrating over momentum variables yields the wanted expressions for the probability density and current. This enables us to define dynamical locality in the sense that the current must vanish in domains where the probability is zero. It is shown that locality exists for a specific class of Hamiltonians only. There exists in this case a preferential polarization of phase space into the ordinary configuration and linear momentum spaces.

2. Geometrical framework and Wigner-functions

The phase spaces E of the dynamical systems considered in this paper are affine symplectic manifolds homeomorphic to $\mathbb{R}^{2n}$. Points of E are labelled by their position vectors $x = (x^1, \ldots, x^{2n})$. The components x^μ form a frame of *linear* coordinates, which are moreover *canonical* when the symplectic 2-form l of E reads explicitly

$$l(x, y) = x \cdot Ly, \qquad L = \begin{pmatrix} 0 & -\mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix} \tag{2.1}$$

The inverse $\Lambda = L^{-1} = -L$ of L will be frequently used. We shall only need Galilean polarizations of E; they split up E into the particular isotropic subspaces

$$E_q = \tfrac{1}{2}(\mathbb{1} + T)E, \tag{2.2}$$

the configuration space, and

$$E_p = \tfrac{1}{2}(\mathbb{1} - T)E, \tag{2.3}$$

the momentum space. Accordingly, only Galilean canonical coordinates will be used in order to have $T = \begin{pmatrix} \mathbb{1}_n & 0 \\ 0 & -\mathbb{1}_n \end{pmatrix}$; in other words, $x^k = q^k$ $(k = 1 \cdots n)$ is a true position coordinate and $x^{k+n} = p_k$ $(k = 1 \cdots n)$ a true linear momentum.

In this framework, the Wigner–Weyl isomorphism Φ maps the set $\mathscr{A}$ of operators on $\mathscr{H} = L^2(E_q)$ into the set M of functions on E:

$$\Phi : F \to f = \Phi[F]$$

$$f(x) = f(q, p) = \int_{E_q} d^n q' e^{(i/\hbar)q' \cdot p} \langle q - \tfrac{1}{2}q' |F| q + \tfrac{1}{2}q' \rangle. \tag{2.4}$$

$f = \Phi[F]$ is the Wigner function of the operator F, and $w = \Phi[|\Psi\rangle\langle\Psi|]$ the Wigner function of the pure state $|\Psi\rangle$. The term function must naturally be taken in a large sense since many images of operators are in fact kernels of functionals.

The product of operators goes over to the Moyal product of functions:

$$\Phi[FG] = \Phi[F] \circ \Phi[G] = f \circ g. \tag{2.5}$$

The Moyal product is a particular case of the product [2]

$$(f \underset{\alpha}{\circ} g)(x) = \int_E \frac{d^{2n}y}{(\pi\alpha)^n} \frac{d^{2n}z}{(\pi\alpha)^n} e^{-(2i/\alpha)l(y-x, z-x)} f(y) g(z). \tag{2.6}$$

The parameter α labels the continuous set of all associative, non-commutative, involutive and symplectic-invariant products. The limit $\alpha = 0$ is the ordinary product,

$$f \underset{\alpha}{\circ} g \xrightarrow[\alpha \to 0]{} fg, \tag{2.7}$$

and the *Moyal product* is by definition*)

$$f \circ g = f \underset{\alpha}{\circ} g|_{\alpha = \hbar}. \tag{2.8}$$

The integral of $f \circ g$ over E, whenever it converges, no longer depends on α:

$$\int_E d^{2n}x (f \underset{\alpha}{\circ} g)(x) = \int_E d^{2n}x f(x) g(x). \tag{2.9}$$

* Also called *twisted product.*

The *Moyal bracket* is, up to a factor, the image of the commutator:

$$\{f, g\}_M = \frac{1}{i\hbar}(f \circ g - g \circ f) = \frac{1}{i\hbar}\Phi[[F, G]]. \tag{2.10}$$

In particular cases, Moyal and Poisson brackets coincide; for instance:

$$\{x^\mu, f\}_M = \{x^\mu, f\}_P = \Lambda^{\mu\nu}\frac{\partial f}{\partial x^\nu}. \tag{2.11}$$

For many purposes, it is useful to know the rule

$$x^\mu \circ f = x^\mu f - \frac{\hbar}{2i}\Lambda^{\mu\nu}\frac{\partial}{\partial x^\nu}f, \tag{2.12}$$

which is a direct consequence of the definition (2.6).

3. Current and quasi-current densities

Time dependent states of quantum systems are described in phase space E by functions w_t on E parametrized by the time t. The *quasi-probability* w_t is for each t:

$$\text{real:}\quad w_t^* = w_t \tag{3.1}$$

$$\text{normalized:}\quad \operatorname{Tr} w_t = \int_E \frac{d^{2n}x}{(2\pi\hbar)^n} w_t(x) = 1 \tag{3.2}$$

$$M\text{-positive:}\quad \exists v_t \quad \text{such that} \quad w_t = v_t \circ v_t^*. \tag{3.3}$$

In consequence of (3.2–3) and (2.9) w_t is $L^2(E)$ and satisfies $\operatorname{Tr} w_t \circ w_t \leqslant \operatorname{Tr} w_t$. The equality holds for *pure* states: $w_t \circ w_t = w_t$. Given a Galilean polarization $x = q \oplus p$ of E, the usual *probability density* in configuration space is obtained by

$$\rho_t(q) = \int_{E_p} \frac{d^n p}{(2\pi\hbar)^n} w_t(q, p) \geqslant 0. \tag{3.4}$$

In order to satisfy the conservation law (3.2) one can try to find a vector field $\mathcal{T}_t$ on E whose divergence yields the time derivative of w_t:

$$\partial_t w_t + \nabla \cdot \mathcal{T}_t = 0. \tag{3.5}$$

$\mathcal{T}_t$ is interpreted as a *quasi-current density* and (3.5) is a continuity equation in phase space. For any compact domain $\mathcal{D}$ of E the relation (3.5) implies

$$\int_{\mathcal{D}} d^{2n}x\, \partial_t w_t(x) = \oint_{b\mathcal{D}} d\Sigma \cdot \mathcal{T}_t. \tag{3.6}$$

Whenever the left handside of (3.6) converges uniformly when $\mathcal{D} \to E$, the time derivative ∂_t and the integral are permutable. From (3.2) it follows that this expression vanishes in the limit $\mathcal{D} \to E$. So does the right handside of (3.6) with the consequence that $\mathcal{T}_t$ must be asymptotically a divergence free field.

The *current density* j_t in q-space is obtained from $\mathcal{T}_t$ by

$$j_t^k(q) = \int_{E_p} \frac{d^n p}{(2\pi\hbar)^n} \mathcal{T}_t^k(q, p), \quad k = 1 \cdots n. \tag{3.7}$$

A consequence of (3.5) is the usual continuity equation

$$\partial_t \rho_t + \nabla_q \cdot j_t = 0. \tag{3.8}$$

For a given w_t, new solutions of (3.5) are easily obtained from a known one. For instance

$$\mathcal{T}_t' = \mathcal{T}_t + \Lambda \nabla f, \tag{3.9}$$

where f is any differentiable "potential". But the physical quasi-current has to fulfill further requirements bound to the dynamics of the system. The dynamics are governed by the von Neumann equation

$$\partial_t w_t = \{h, w_t\}_M \stackrel{d}{=} \frac{1}{i\hbar}(h \circ w_t - w_t \circ h) \tag{3.10}$$

and by the equation of motion for the observable x

$$\dot{x} = \{x, h\}_M \equiv \Lambda \nabla h, \tag{3.11}$$

where h is the Hamilton function of the system. For physical reasons, the quasi-current must be linear in the velocity $\dot{x}$ and in the density w_t, these properties persisting after the reduction (3.7). The most general expression is a vector valued functional of both quantities. Before stating the result, let us remark that the integral over E of both sides of (3.10) vanishes whenever $h \circ w_t \in L^1(E)$ because the integral of the Moyal product $h \circ w_t$ is equal to the integral of the ordinary commutative product hw_t.

We prove below that the linear functional of w_t and $\dot{x}$

$$\mathcal{T}_t(x) = \frac{1}{2} \int_{-1}^{1} ds \int_E \frac{d^{2n} y}{(\pi\hbar)^n} \int_E \frac{d^{2n} z}{(\pi\hbar)^n} e^{-(2i/\hbar) l(y,z)} w_t(y + x)(\Lambda \nabla h)(sz + x) \tag{3.12}$$

satisfies the continuity equation (3.5). By performing the integration variable substitution $z \to z' = sz$ and introducing the product $\underset{\alpha}{\circ}$ (2.6) one obtains the more transparent form

$$\mathcal{T}_t = \frac{1}{2\hbar} \int_{-\hbar}^{\hbar} d\alpha w_t \underset{\alpha}{\circ} \Lambda \nabla h = \frac{1}{\hbar} \int_0^{\hbar} d\alpha \tfrac{1}{2}(w_t \underset{\alpha}{\circ} \dot{x} + \dot{x} \underset{\alpha}{\circ} w_t), \tag{3.13}$$

$\dot{x}$ being naturally supposed to satisfy (3.11). It is now obvious that $\mathcal{T}_t$ is a real field on E. In the limit $\hbar \to 0$ the functional (3.13) becomes a function and $\mathcal{T}_t$ coincides with the classical expression:

$$\mathcal{T}_t \xrightarrow[\hbar \to 0]{} \lim_{\hbar = 0} \frac{1}{\hbar} \int_0^{\hbar} d\alpha \tfrac{1}{2}(w_t \underset{\alpha}{\circ} \dot{x} + \dot{x} \underset{\alpha}{\circ} w_t) = \tfrac{1}{2}(w_t \underset{\alpha}{\circ} \dot{x} + \dot{x} \underset{\alpha}{\circ} w_t)\big|_{\alpha=0} = w_t \dot{x}. \tag{3.14}$$

The last equality holds because the product $\underset{\alpha}{\circ}$ reduces to the ordinary product of functions for $\alpha = 0$.

The proof of our assertion follows directly from the mathematical identity

$$\nabla \cdot \mathcal{T}_t = \{w_t, h\}_M \tag{3.15}$$

which holds when $\mathcal{T}_t$ is defined by (3.12). Indeed, (3.15) is equivalent to the continuity equation if w_t satisfies the state equation (3.10). Proceeding from (3.13) one gets

$$\begin{aligned}\nabla \cdot \mathcal{T}_t(x) &= \frac{1}{2\hbar}\int_{-\hbar}^{\hbar} d\alpha \int_{E\times E} \frac{d^{2n}y\, d^{2n}z}{(\pi\alpha)^{2n}}\, e^{-(2i/\alpha)l(y,z)}\nabla_x \cdot (w_t(x+y)\Lambda\nabla h(x+z))\\ &= \frac{1}{2\hbar}\int_{-\hbar}^{\hbar} d\alpha \left[\nabla_{x'} \cdot \Lambda\nabla_x \int_{E\times E} \frac{d^{2n}y\, d^{2n}z}{(\pi\alpha)^{2n}}\, e^{-(2i/\alpha)l(y,z)} w_t(x'+y)h(x+z)\right]_{x'=x}.\end{aligned}$$

The last equality is a consequence of Λ being skew-symmetric. The substitutions $z \to z' = z + x$ and $y \to y' = y + x$ yield

$$\begin{aligned}\nabla \cdot \mathcal{T}_t(x) &= \frac{1}{2\hbar}\int_{-\hbar}^{\hbar} d\alpha \int_{E\times E} \frac{d^{2n}y'\, d^{2n}z'}{(\pi\alpha)^{2n}}\, w_t(y')h(z')\\ &\quad\times [\nabla_{x'} \cdot \Lambda\nabla_x\, e^{-(2i/\alpha)l(y'-x',z'-x)}]_{x'=x}\end{aligned}$$

By means of the identities

$$\begin{aligned}&\nabla_{x'} \cdot \Lambda\nabla_x e^{-(2i/\alpha)l(y'-x',z'-x)}\big|_{x'=x}\\ &= \nabla_{y'}\, \Lambda\nabla_{z'} e^{-(2i/\alpha)l(y'-x,z'-x)}\\ &= \left(\frac{2}{\alpha}\right)^2 l(y'-x,\, z'-x)e^{-(2i/\alpha)l(y'-x,z'-x)} + \frac{4ni}{\alpha}\, e^{-(2i/\alpha)l(y'-x,z'-x)}\\ &= \frac{2}{i}\,\alpha^{2n}\frac{\partial}{\partial\alpha}\,\alpha^{-2n}e^{-(2i/\alpha)l(y'-x,z'-x)}\end{aligned}$$

one finally obtains

$$\begin{aligned}\nabla \cdot \mathcal{T}_t(x) &= \frac{1}{i\hbar}\int_{-\hbar}^{\hbar} d\alpha \frac{\partial}{\partial\alpha}\int_{E\times E} \frac{d^{2n}y'\, d^{2n}z'}{(\pi\alpha)^{2n}}\, e^{-(2i/\alpha)l(y'-x,z'-x)} w_t(y')h(z')\\ &= \frac{1}{i\hbar}(w_t \circ h)(x)\big|_{\alpha=-\hbar}^{\alpha=\hbar} = \frac{1}{i\hbar}(w_t \circ h - h \circ w_t)(x)\\ &= \{w_t, h\}_M(x).\end{aligned}$$

This proves (3.15).

The current density in q-space derived from the quasi-current (3.12) reads after some computations

$$j_t^k(q) = \frac{1}{2}\int_{-1}^{1} ds \int_{E_q} d^nq' f_t(q+(s-1)q',\, q+(s+1)q')\chi^k(q+sq',\, q') \tag{3.16}$$

where

$$\chi^k(q+sq', q') = \int_{E_p} \frac{d^n p}{(\pi\hbar)^n} e^{i(2/\hbar)p\cdot q'} \frac{\partial h}{\partial p_k}(q+sq', p), \quad k = 1 \cdots n. \tag{3.17}$$

The function f_t is related to w_t by

$$w_t(q, p) = 2^n \int_{E_q} d^n q' e^{-i(2/\hbar)p\cdot q'} f_t(q-q', q+q') \tag{3.18}$$

and also describes the state of the system. It can always be written as

$$f_t(q_1, q_2) = \int_{E_q} d^n q_3 g_t(q_1, q_3) g_t(q_2, q_3)^* \tag{3.19}$$

and must fulfill the normalization condition

$$\int_{E_q} d^n q f(q, q) = \int_{E_q \times E_q} d^n q \, d^n q' \, |g_t(q, q')|^2 = 1. \tag{3.20}$$

In this way, the parametrization (3.18) yields a function w_t which satisfies automatically the three requirements (3.1–3).

For pure quantum states, $w_t \circ w_t = w_t$, the kernel f_t becomes separable

$$f_t(q, q') = \psi(q)\psi(q')^*. \tag{3.21}$$

The factor ψ_t is the Schrödinger wave function. In any case the density of probability is equal to

$$\rho_t(q) = f_t(q, q). \tag{3.22}$$

If ρ_t vanishes at $q = q_0$, then

$$f(q', q_0) = f(q_0, q') = 0 \qquad \forall q' \tag{3.23}$$

because of the special form (3.19) of f_t.

Remark. The validity of (3.15) is in fact based on a general property of the Moyal bracket. The function $\{a, b\}_M(x)$ remains unchanged if one adds constants to the functions a and b because $\alpha \circ b = \alpha b$ when α is a constant. This suggests that the Moyal bracket may be written as a functional of the first derivatives of a and b (supposed differentiable). One finds indeed that

$$\{a, b\}_M(x) = \int_{E\times E} \frac{d^{2n}y \, d^{2n}z}{(\pi\hbar)^{2n}} K\left(\frac{2}{\hbar} l(y-x, z-x)\right) \nabla a(y) \cdot \Lambda \nabla b(z) \tag{3.24}$$

with

$$K(\lambda) = \int_0^1 dt t^{-2n} \cos\frac{\lambda}{t}. \tag{3.25}$$

The kernel $k(y, z) = K(2\hbar^{-1} l(y, z))$ is obviously invariant under homogeneous symplectic transformations and satisfies the "d'Alembert" equation in $E \times E$

$$\nabla_y \cdot \Lambda \nabla_z k(y, z) = -\frac{2}{\hbar} \sin \frac{2}{\hbar} l(y, z). \tag{3.26}$$

The following properties of k will be useful in the next section

$$\int d^{2n} z k(y, z) = \delta^{(2n)}(y) \tag{3.27}$$

$$\int d^{2n} z z^{\nu_1} \cdots z^{\nu_m} k(y, z) = \tfrac{1}{2}(1+(-1)^m) \frac{1}{m+1} \left(\frac{\hbar}{2i}\right)^m (\Lambda\nabla)^{\nu_1} \cdots (\Lambda\nabla)^{\nu_m} \delta^{(2n)}(y) \nu_i = 1 \cdots 2n. \tag{3.28}$$

They indicate that (3.24) is valid for a larger class of pairs a, b than merely w_t, h pairs.

4. Locality and E-locality

The notions of locality introduced here are dynamical properties which allow a classification of quantum systems and their Hamilton functions. They are meaningful for localizable systems only, like those we consider here. (The characteristic functions of compact domains of configuration space form a complete set of commuting projectors [3]). Definition of locality:

"A system is said to be *local in configuration space*, or simply *local*, if for each physical state the current density $j_t(q)$ vanishes in every point q_0 where the density $\rho_t(q)$ vanishes itself:

$$\rho_t(q_0) = 0 \Rightarrow j_t(q_0) = 0." \tag{4.1}$$

A local system never makes "ghost apparitions" in successive disconnected domains.
Definition of E-locality:

"A system is said to be *local in phase space*, or simply *E-local*, if the implication

$$w_t(x_0) = 0 \Rightarrow \mathcal{T}_t(x_0) = 0 \tag{4.2}$$

holds for every physical state."

The meaning of E-locality is certainly less concrete than that of locality. Nevertheless, it leads to an interesting comparison with classical systems as we shall see.

Assuming that the quasi-current $\mathcal{T}_t$ is defined as in (3.12) the property of being local or E-local depends only on the form of the Hamiltonian h. More precisely:

Theorem 1. *A system is E-local if and only if its Hamilton function is a polynomial of degree 2 at most in the linear canonical coordinates x^μ.*

Theorem 2. *A system is local (in configuration space E_q) if and only if its Hamilton function $h(q, p)$ is of degree 2 at most in the linear momentum p.*

Proof of Theorem 1

The quasi-current written in condensed form reads

$$\mathcal{T}_t(x) = \int_E \frac{d^{2n}y}{(\pi\hbar)^n} w_t(x+y)\Delta_h(y, x)$$

with

$$\Delta_h(y, x) = \int_E \frac{d^{2n}z}{(\pi\hbar)^n} K\left(\frac{2}{\hbar} l(y, z)\right)\Lambda\nabla h(x+z).$$

E-locality requires first that the domain of the kernel Δ_h be punctual. With the help of (3.27–28) one sees that h must be polynomial to satisfy this condition. Secondly, $\mathcal{T}_t$ cannot depend on derivatives of w_t, which are not supposed to vanish in x_0. This limits the degree of h to 2. QED.

Proof of Theorem 2

From the relations (3.22) and (3.23) one sees that the current $\mathcal{T}_t$ vanishes at the same places as ρ_t only when the kernel χ (3.17) has a punctual domain. This implies that h be polynomial in p. Supposing this is the case, χ is a finite sum of $\delta^{(n)}(q')$ and its derivatives, multiplied by some functions of $q+sq'$, the order of the highest derivative being equal to the degree of h minus one. Integrating (3.16) by parts, one can easily integrate over q', and the result is a j_t equal to a sum over partial derivatives of f_t. The terms with mixed derivatives like $(\partial^2/\partial q\, \partial q')f(q, q')|_{q'=q}$ do not systematically vanish when (3.23) holds. Therefore, $j_t(q_0) = 0$ only when h is a second degree polynomial in p at most. QED.

E-locality clearly implies locality. E-locality is a specific property of linear mechanics, which is characterized by Hamiltonians of degree 2 at most in the linear coordinates x^μ (free motion, oscillators, ...). The definition of E-locality does not depend on a polarization of E into isotropic manifolds and is thus invariant under any coordinate transformation. The class of E-local systems contains exactly all systems which have the same classical and quantum dynamics [4]. For these systems, the only difference between the two theories lies in the type of initial states $w_0(x)$ that one is allowed to postulate: A classical w_0 must be a positive function, interpreted as a true density of probability on E. A quantum w_0 must be an M-positive function; it satisfies the uncertainty relations. The quasi-current of an E-local system is exactly given by the classical expression

$$\mathcal{T}_t = \dot{x} w_t. \tag{4.3}$$

On the other hand, the locality restricts the p dependence of h only. Its definition is only invariant under coordinate transformations in q-space, enhancing the physical significance of this isotropic sub-manifold of E.

In conclusion, the most general Hamiltonian allowed by locality has the form

$$\begin{aligned} h(x) &= \sum_{k,l=1}^{n} \Gamma^{kl}(q)(p_k + A_k(q))(p_l + A_l(q)) + V(q) \\ &\equiv \sum_{k,l=1}^{n} (p_k + A_k)\circ\Gamma^{kl}\circ(p_l + A_l) + V(q) - \left(\frac{\hbar}{2}\right)^2 \partial_k\partial_l\Gamma^{kl}(q). \end{aligned} \tag{4.4}$$

It is remarkable that this result coincides precisely for one particle with the most general Hamiltonian having the full passive Galilean invariance [5, 6]. The current density of a local system reduces to the simple form

$$j_t^k(q) = \int_{E_p} \frac{d^n p}{(2\pi\hbar)^n} \dot{q}^k(q, p) w_t(q, p)$$
$$= \left(\frac{\partial h}{\partial p_k}(q, 0) - \frac{\hbar}{2i} \sum_{l=1}^{n} \frac{\partial^2 h}{\partial p_k\, \partial p_l}(q, 0) \frac{\partial}{\partial q'^l}\right) f_t(q-q', q+q')\big|_{q'=q}. \tag{4.5}$$

5. Examples of probability current densities

a) *Schrödinger equation with magnetic field and local potential*

The Hamilton-function is

$$h = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}(\vec{q})\right)^2 + V(\vec{q})$$
$$\equiv \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right) \circ \left(\vec{p} - \frac{e}{c}\vec{A}\right) + V. \tag{5.1}$$

Equation (4.5) together with (3.21) gives for pure states

$$j_t(\vec{q}) = \left(-\frac{e}{mc}\vec{A} - \frac{\hbar}{2im}\vec{\nabla}_{q'}\right)\psi(\vec{q} - \vec{q}')\psi^*(\vec{q} + \vec{q}')\big|_{\vec{q}'=\vec{q}}$$

or

$$j_t(\vec{q}) = \frac{\hbar}{2im}(\psi_t^*(\vec{q})\,\vec{\nabla}\psi_t(\vec{q}) - \psi_t(\vec{q})\,\vec{\nabla}\psi_t^*(\vec{q})) - \frac{e}{mc}\vec{A}(\vec{q})\,|\psi_t(\vec{q})|^2. \tag{5.2}$$

This is the well known form of the probability current. Our expression (3.16) is obviously a generalization of this case.

b) *Kisslinger equation*

The Kisslinger equation [7] is a traditional tool used for the phenomenological description of the pion-nucleus interaction. This is a Schrödinger equation modified by the introduction of an effective mass depending on the nuclear density. The Hamilton-function has the form

$$h(\vec{q}, \vec{p}) = \vec{p} \circ \frac{1}{2m} \circ \vec{p} + V(\vec{q}) \equiv \frac{1}{2m(\vec{q})}\vec{p}^2 + V(\vec{q}) + \frac{\hbar}{8}\Delta_q \frac{1}{m(\vec{q})}. \tag{5.3}$$

The current density reads

$$j_t(\vec{q}) = \frac{\hbar}{2im(\vec{q})}(\psi_t^*(\vec{q})\,\vec{\nabla}\psi_t(\vec{q}) - \psi_t(\vec{q})\,\vec{\nabla}\psi_t^*(\vec{q})). \tag{5.4}$$

It is a trivial extension of the previous case. The customary interpretation of $|\psi|^2$ as the probability density is preserved here, in contrast to propositions by D. Pai and E. Vogt [8].

c) *One band approximation*

In solid state physics, the effective mass is a function of p and does not depend on position. One expects a periodic function of p in the description of a single band. As an illustration, let us consider the one-dimensional model

$$h(q, p) = \frac{p_0^2}{2m_0}\left(\sin\frac{p}{p_0}\right)^2 + V(q). \tag{5.5}$$

The system is no longer local and we must start from expressions (3.16–17) to calculate the current which reads finally

$$\begin{aligned} j_t(q) = \frac{1}{2}\int_{-1}^{1} ds \frac{p_0}{2im_0}\left\{\psi_t^*\left(q-(s+1)\frac{\hbar}{p_0}\right)\psi_t\left(q-(s-1)\frac{\hbar}{p_0}\right)\right. \\ \left. -\psi_t^*\left(q+(s+1)\frac{\hbar}{p_0}\right)\psi_t\left(q+(s-1)\frac{\hbar}{p_0}\right)\right\}. \end{aligned} \tag{5.6}$$

The non-locality extends as far as $\Delta q = \pm 2\hbar p_0^{-1}$. The non-local relation (5.6) between j and ψ is not a particular case of (4.5). The physical meaning of the quantities j, $\dot{q}$ and $|\psi|^2$ will need some more thought to be really understood.

d) *Non-local potential contribution to the current*

The contributions to the current (3.16) coming from different terms of h can be isolated by linearity. In nuclear physics one frequently makes use of so called separable potentials as an approximation of non-local ones. The Wigner images in phase space of such potentials are complicated functions of q and p simultaneously. A simple example is given by the projector onto the ground-state of the harmonic oscillator. A potential proportional to this projector reads

$$V(q, p) = \alpha 2e^{-(1/\hbar\omega)(kq^2+(1/m)p^2)}, \qquad \omega^2 = \frac{k}{m}, \tag{5.7}$$

where α is its strength. Its contribution to the current is easily calculated from (3.16–17):

$$\begin{aligned} j_{\text{pot}}(q) = -i\alpha\sqrt{\frac{m\omega}{\pi\hbar^3}} \\ \times\int_{-1}^{1} ds\int_{-\infty}^{\infty} dq'q'e^{-(m\omega/\hbar)(q'^2+(q+sq')^2)}\psi(q+(s-1)q')\psi^*(q+(s+1)q'). \end{aligned} \tag{5.8}$$

This part of the current vanishes outside the interaction region.

Note added in proof. We learned recently of the works of R. E. Collins [9] on a related subject. His starting point is different and we think that the time evolution defined by the potential (5.7) above exhibits a counter example to his conclusions.

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