On regular external fields in quantum electrodynamics

Autor(en): **Nenciu, G. / Scharf, G.**

Objekttyp: **Article**

Zeitschrift: **Helvetica Physica Acta**

Band (Jahr): **51 (1978)**

Heft 3

PDF erstellt am: **22.09.2024**

Persistenter Link: https://doi.org/10.5169/seals-114960

Helvetica Physica Acta, Vol. 51 (1978), Birkhäuser Verlag, Basel

On regular external fields in quantum electrodynamics

by **G. Nenciu**[1]) and **G. Scharf**

Institut für Theoretische Physik der Universität Zürich, Schönberggasse 9, Ch-8001 Zürich

(28. III. 1978)

Abstract. Regular external fields are defined by the property that the electron–positron emission and absorption operators, in the presence of the classical electromagnetic field, realize a Fock representation of the anti-commutation relations on the Fock space of the free field. Explicit necessary and sufficient conditions for static regular external potentials are given. There exists only regular static electric fields, no static magnetic field is regular in the above sense.

1. Introduction

Regular external electromagnetic fields are defined by the property that the Dirac equation, with these external fields, allows a second quantization within the ordinary Fock space of the free Dirac field. More precisely, the electron–positron emission and absorption operators, in the presence of the classical external field (dressed operators), realize a Fock representation of the anti-commutation relations on the Fock space of the free field. A necessary and sufficient criterion for regularity has been derived in [1]: A static external field is regular if and only if the one-particle Dirac operator

$$H = -i \sum_{k=1}^{3} \alpha_k \frac{\partial}{\partial x_k} + m\beta + V(\mathbf{x}) - \boldsymbol{\alpha} \cdot \mathbf{A}(\mathbf{x}) \tag{1}$$

has the property that $P_+ - P_+^0$ is a Hilbert–Schmidt operator, where P_+ and P_+^0 are the spectral projections on the positive parts of the spectra of H and the free Dirac operator H_0, respectively. The investigation of this basic criterion by means of perturbation methods [1] leads to the very natural conjecture that a static external field is regular iff it is purely electric ($\mathbf{A} = 0$) and the Fourier transformed potential satisfies

$$\int d^3p \int d^3q \frac{|\hat{V}(\mathbf{p} - \mathbf{q})|^2}{(E_p + E_q)^2} \left(1 - \frac{\mathbf{p} \cdot \mathbf{q} + m^2}{E_p E_q}\right) < \infty. \tag{1.2}$$

[1]) On leave from the Laboratory of Fundamental Physics, Institute of Nuclear Physics and Engineering, Bucharest, Romania.

It has been shown in [1] (for another proof see Theorem 2 below) that this condition (1.2) is equivalent to

$$\int d^3p \frac{|\mathbf{p}|^2}{1+|\mathbf{p}|} |\hat{V}(\mathbf{p})|^2 < \infty. \tag{1.3}$$

The class of potentials $V(\mathbf{x})$ satisfying (1.3) will be denoted by A_0 in the following.

In the present paper we return once more to this problem of characterization of regular static potentials. The proof of the conjecture (1.2) seems to be rather complicated. However, we are able to approach the condition (1.3) from both sides arbitrarily close. We will prove the following necessary condition for regularity (Theorem 2, next section)

$$\int d^3p \frac{|\mathbf{p}|^2}{1+|\mathbf{p}|^{1+\varepsilon}} |\hat{V}(\mathbf{p})|^2 < \frac{C}{\varepsilon} \tag{1.4}$$

for any $\varepsilon > 0$, where C is independent of ε. In particular no static magnetic field is regular (Theorem 1), which from the physical point of view is surprising. Condition (1.4) is weaker than (1.3) in the ultraviolet limit $p \to \infty$. A stronger necessary condition in this region is

$$\int_{n_1 N \leq p \leq n_2 N} d^3p |\mathbf{p}| |\hat{V}(\mathbf{p})|^2 < C \tag{1.5}$$

for any $N \geq 0$ and fixed $n_2 > n_1 > 0$, where C is independent of N (Theorem 3).

Concerning a sufficient condition, we prove in Section 3 (Theorem 4) that V is regular if

$$\int d^3p \frac{|\mathbf{p}|^2}{1+|\mathbf{p}|^{1-\varepsilon}} |\hat{V}(\mathbf{p})|^2 < \infty \tag{1.6}$$

for some $\varepsilon > 0$. This class of potentials denoted by A_ε is smaller than A_0. We have a proof of regularity for potentials satisfying

$$\int d^3p \frac{|\mathbf{p}|^2 \log(2+|\mathbf{p}|)}{1+|\mathbf{p}|} |\hat{V}(\mathbf{p})|^2 < \infty. \tag{1.7}$$

But since this is not yet the final answer, we do not give it here. In addition, we know that the first *two* orders in the perturbation expansion of $P_+ - P_+^0$ are Hilbert–Schmidt operators for $V \in A_0$.

2. Necessary conditions for regular potentials

We consider the Dirac hamiltonian

$$H = H_0 + H_1 \tag{2.1}$$

with

$$H_1 = V(\mathbf{x}) - \boldsymbol{\alpha}\cdot\mathbf{A}(\mathbf{x}). \tag{2.2}$$

Let us assume that the interaction part H_1 to be H_0-bounded with H_0-bound < 1 in order to have a self-adjoint hamiltonian H by the Rellich–Kato method. This assumption is not essential and can be weakened for instance to H_0-form boundedness.

We now require that

$$P_+ - P_+^0 \in \text{H.S.} \tag{2.3}$$

is a Hilbert–Schmidt (H.S.) operator, and look for the most restrictive condition this implies on H_1. The condition (2.3) is equivalent to the two conditions (see [1], Section 4)

$$P_+^0 P_- \in \text{H.S.}, \qquad P_+ P_-^0 \in \text{H.S.} \tag{2.4}$$

Then we have

Theorem 1. *Let H_1 be H_0-bounded with H_0-bound < 1 and such that $P_+^0 P_-$ and $P_+ P_-^0$ are Hilbert–Schmidt operators and 0 is not an eigenvalue of H. Then the vector potential $\mathbf{A}(\mathbf{x})$ must be 0 and the electrostatic potential $V(\mathbf{x})$ satisfies*

$$\int d^3p \int d^3q \frac{|\hat{V}(\mathbf{p} - \mathbf{q})|^2}{(E_p + E_q)^{2+\varepsilon}} \left(1 - \frac{\mathbf{p}\cdot\mathbf{q} + m^2}{E_p E_q}\right) < \frac{C}{\varepsilon} \tag{2.5}$$

for any $\varepsilon > 0$, where C is a constant independent of ε.

Remarks. If 0 is an eigenvalue of H the projections $P_\pm$ have to be defined more carefully (for a discussion of this point see [2], Section 3). It has been shown in [1] that (1.2) implies indeed that V is H_0-bounded with arbitrarily small H_0-bound.

Proof. 1. *Let*

$$R(\lambda) = (H - \lambda)^{-1}, \qquad R_0(\lambda) = (H_0 - \lambda)^{-1} \tag{2.6}$$

be the resolvents of H and H_0, respectively. Then

$$P_+^0 R(i\eta) P_-^0 = P_+^0 P_- R(i\eta) P_-^0 + P_+^0 R(i\eta) P_+ P_-^0 \in \text{H.S.} \tag{2.7}$$

is a Hilbert–Schmidt operator and

$$\|P_+^0 R(i\eta) P_-^0\|_{\text{H.S.}} \leq \frac{\text{const}}{1 + \eta} \quad \text{for all } \eta \geq 0. \tag{2.8}$$

Using

$$R = R_0 - R_0 H_1 R \tag{2.9}$$

we get

$$P_+^0 R P_-^0 = P_+^0 R_0 H_1 P_+^0 R_0 H_1 R P_-^0 + P_+^0 R_0 H_1 R_0 P_-^0 H_1 R P_-^0 - P_+^0 R_0 H_1 R_0 P_-^0 \tag{2.10}$$

and therefore

$$P_+^0 R_0 H_1 R_0 P_-^0 (P_-^0 H_1 R P_-^0 - 1) = P_+^0 (1 + R_0 H_1) P_+^0 R P_-^0 \tag{2.11}$$

must be a Hilbert–Schmidt operator because $R_0(i\eta) H_1$ is bounded. Since $P_-^0 H_1 R(i\eta) P_-^0 - 1$ has an inverse uniformly bounded for $\eta \geq 0$ (note that $P_-^0 H_1 R(i\eta) P_-^0 f = f$ implies $f = 0$), we conclude

$$\|P_+^0 R_0(i\eta) H_1 R_0(i\eta) P_-^0\|_{\text{H.S.}}^2 \leq \frac{\text{const}}{\eta^2} \tag{2.12}$$

where the constant is independent of η.

The Hilbert–Schmidt norm (2.12) can be expressed in terms of the potentials as follows

$$\|P_+^0 R_0(i\eta) H_1 R_0(i\eta) P_-^0\|_{\text{H.S.}}^2 = \int d^3p \int d^3q \frac{1}{(E_p^2 + \eta^2)(E_q^2 + \eta^2)}$$
$$\times \left\{ |\hat{V}(\mathbf{p} - \mathbf{q})|^2 \left(1 - \frac{\mathbf{p}\cdot\mathbf{q} + m^2}{E_p E_q}\right) + |\hat{\mathbf{A}}(\mathbf{p} - \mathbf{q})|^2 \left(1 + \frac{\mathbf{p}\cdot\mathbf{q} + m^2}{E_p E_q}\right) - 2\,\mathrm{Re}\,\frac{\mathbf{p}\cdot\hat{\mathbf{A}}(\mathbf{p} - \mathbf{q})\mathbf{q}\cdot\hat{\mathbf{A}}(\mathbf{p} - \mathbf{q})^*}{E_p E_q} - 2\,\mathrm{Re}\,\hat{V}(\mathbf{p} - \mathbf{q})\left(\frac{\mathbf{p}\cdot\hat{\mathbf{A}}(\mathbf{p} - \mathbf{q})^*}{E_p} - \frac{\mathbf{q}\cdot\hat{\mathbf{A}}(\mathbf{p} - \mathbf{q})^*}{E_q}\right)\right\}. \tag{2.13}$$

2. Let us examine the magnetic terms at first. Collecting the terms involving the x-component $|\hat{A}_1(p - q)|^2$ we obtain

$$J_1 = \int d^3k_1 B_1(\mathbf{k}_1)|\hat{A}_1(\mathbf{k}_1)|^2 \tag{2.14}$$

where

$$B_1(\mathbf{k}_1) = 32\pi \int_0^\infty dk_2 \int_{-1}^{+1} dz \frac{k_2^2}{(k_2^2 + k_1^2 + 4m^2 + 4\eta^2)^2 - 4k_1^2k_2^2z^2} \times \left(1 + \frac{k_2^2 - k_1^2 + 4m^2}{\sqrt{(k_2^2 + k_1^2 + 4m^2)^2 - 4k_1^2k_2^2z^2}} - 2\frac{k_{2x}^2 - k_{1x}^2}{\sqrt{(k_2^2 + k_1^2 + 4m^2)^2 - 4k_1^2k_2^2z^2}}\right) \tag{2.15}$$

and the new variables of integration

$$\mathbf{p} - \mathbf{q} = \mathbf{k}_1, \qquad \mathbf{p} + \mathbf{q} = \mathbf{k}_2 \tag{2.16}$$

have been used. Let us consider the spherically symmetric part $B_0(|\mathbf{k}_1|)$ of (2.15)

$$B_0(k_1) = 32\pi \int_0^\infty dk_2 \int_{-1}^{+1} dz \frac{k_2^2}{(k_2^2 + k_1^2 + 4m^2 + 4\eta^2)^2 - 4k_1^2k_2^2z^2} \times \left(1 + \frac{\frac{1}{3}k_2^2 - k_1^2 + 4m^2}{\sqrt{(k_2^2 + k_1^2 + 4m^2)^2 - 4k_1^2k_2^2z^2}}\right). \tag{2.17}$$

To estimate this from below, we split the k_2-integral as follows

$$B_0(k_1) = 32\pi\left(\int_0^{k_3} \cdots + \int_{k_3}^\infty \cdots\right) \stackrel{\text{def}}{=} B_{01} + B_{02}$$

with

$$k_3 = \sqrt{3(k_1^2 - 4m^2)} \quad \text{if } k_1 > 2m$$
$$= \quad 0 \qquad\qquad \text{if } k_1 \le 2m. \tag{2.18}$$

For B_{02} we get

$$B_{02}(k_1) \ge 32\pi \int_{k_3}^{\infty} dk_2 \int_{-1}^{+1} dz \frac{k_2^2}{(k_2^2 + d^2)^2}$$
$$= 64\pi \left(\frac{\pi}{4d} - \frac{1}{2d} \operatorname{arc\,tg} \frac{k_3}{d} + \frac{1}{2} \frac{k_3}{k_3^2 + d^2} \right) \tag{2.19}$$

where

$$d^2 = k_1^2 + 4m^2 + 4\eta^2.$$

Since

$$\frac{k_3}{d} \le \sqrt{3}, \qquad \operatorname{arc\,tg} \frac{k_3}{d} \le \frac{\pi}{3},$$

we have

$$B_{02}(k_1) \ge \frac{16}{3} \frac{\pi^2}{\sqrt{k_1^2 + 4m^2 + 4\eta^2}}. \tag{2.20}$$

This leads to the following contribution to J_1 (2.14)

$$J_{12} \ge \frac{16}{3} \pi^2 \int_{|\mathbf{k}|^2 \le 4m^2 + 4\eta^2} d^3k_1 \frac{|\hat{A}_1(\mathbf{k}_1)|^2}{\sqrt{k_1^2 + 4m^2 + 4\eta^2}}$$
$$\ge \frac{16}{3} \frac{\pi^2}{\sqrt{8\eta^2 + 8m^2}} \int_{k_1^2 \le 4m^2 + 4\eta^2} d^3k_1 |\hat{A}_1(\mathbf{k}_1)|^2 \tag{2.21}$$

which decreases only as η^{-1} for large η in contradiction to (2.12). The remaining integral B_{01} over the *finite* interval $[0, k_3]$ gives a contribution $0(\eta^{-4})$, as can be easily estimated, which does not compensate (2.21)

Finally let us consider a typical non-diagonal term involving

$$p_1 \hat{A}_1(\mathbf{p} - \mathbf{q}) q_2 \hat{A}_2(\mathbf{p} - \mathbf{q}) = \tfrac{1}{4}(k_{1x} + k_{2x})(k_{2y} - k_{1y}) \cdot \hat{A}_1(\mathbf{k}_1) \hat{A}_2(\mathbf{k}_1)^*. \tag{2.22}$$

Integrating with respect to $\mathbf{k}_2$, the only term which contributes is of the form $k_{1x} k_{1y} B_2(k_1) \hat{A}_1(\mathbf{k}_1) \hat{\mathbf{A}}_2(\mathbf{k}_1)^*$ where

$$B_2(k_1) = \int_0^{\infty} dk_2 \int_{-1}^{+1} dz \frac{k_2^2}{(k_2^2 + k_1^2 + 4m^2 + 4\eta^2)^2 - 4k_1^2 k_2^2 z^2}$$
$$\times \frac{1}{\sqrt{(k_2^2 + k_1^2 + 4m^2)^2 - 4k_1^2 k_2^2 z^2}}. \tag{2.23}$$

Estimating from above

$$B_2(k_1) < 2 \int_0^{\infty} dk_2 \frac{k_2^2}{[(k_2^2 - k_1^2)^2 + (4\eta^2 + 4m^2)^2] |k_2^2 - k_1^2|}, \tag{2.24}$$

we easily find this contribution to be $0(\eta^{-3})$. Consequently, there is no way to compensate the term $0(\eta^{-1})$ (2.21). Therefore we must have $\hat{A}_1(\mathbf{k}) = 0$ (almost everywhere). Since the same arguments apply to $\hat{A}_2$, $\hat{A}_3$, the vector potential must be 0.

3. The electrostatic potential $\hat{V}$ must satisfy the condition (2.12)

$$\int d^3p \int d^3q \frac{\eta^2}{(E_p^2 + \eta^2)(E_q^2 + \eta^2)} |\hat{V}(\mathbf{p} - \mathbf{q})|^2 \left(1 - \frac{\mathbf{p}\cdot\mathbf{q} + m^2}{E_p E_q}\right) \leq \text{const} \quad (2.25)$$

uniformly in η. We multiply (2.12) by $\eta^{-1-\varepsilon}$ and integrate over η from 1 to ∞. Let us consider the integral

$$J = \int_1^\infty d\eta \frac{\eta^{1-\varepsilon}}{(\eta^2 + E_p^2)(\eta^2 + E_q^2)}. \quad (2.26)$$

For $E_p > E_q > 1$ we have the simple estimates

$$J > \int_1^{2E_p} \cdots > \frac{1}{(2E_p)^\varepsilon} \int_1^{2E_p} d\eta \frac{\eta}{(\eta^2 + E_p^2)^2}$$

$$= \frac{1}{(2E_p)^\varepsilon} \frac{1}{2}\left(\frac{1}{E_p^2 + 1} - \frac{1}{5E_p^2}\right) > \frac{\text{const}}{E_p^{2+\varepsilon}}. \quad (2.27)$$

For $E_q > E_p > 1$ we get in the same way

$$J > \frac{\text{const}}{E_q^{2+\varepsilon}} \quad (2.28)$$

and for arbitrary E_p, E_q

$$J > \frac{\text{const}}{(E_p + E_q)^{2+\varepsilon}}. \quad (2.29)$$

Using this estimate in the integrated form of (2.12) we finally obtain (2.5)

$$\int d^3p \int d^3q \frac{|\hat{V}(\mathbf{p} - \mathbf{q})|^2}{(E_p + E_q)^{2+\varepsilon}} \left(1 - \frac{\mathbf{p}\cdot\mathbf{q} + m^2}{E_p E_q}\right) < C \int_1^\infty \frac{d\eta}{\eta^{1+\varepsilon}} = \frac{C}{\varepsilon}. \quad (2.30)$$

The necessary condition (2.5) is equivalent to the following more explicit condition

Theorem 2. *If $\hat{V}$ is a regular electrostatic potential, it satisfies*

$$\int d^3p \frac{p^2}{1 + p^{1+\varepsilon}} |\hat{V}(\mathbf{p})|^2 < \frac{C}{\varepsilon} \quad (2.31)$$

for any $\varepsilon > 0$ with some constant C independent of ε.

Proof. As in the second part of the proof of Theorem 1, we transform the condition (2.5) using the integration variables (2.16)

$$\int d^3k_1 B(k_1) |\hat{V}(\mathbf{k}_1)|^2 < \frac{C}{\varepsilon} \quad (2.32)$$

with

$$B(k_1) = 2\pi \int_0^\infty dk_2 \int_{-1}^{+1} dz \frac{k_2^2}{(\frac{1}{2}\sqrt{a+bz} + \frac{1}{2}\sqrt{a-bz})^{2+\varepsilon}} \left(1 - \frac{k_2^2 - k_1^2 + 4m^2}{\sqrt{a^2 - b^2 z^2}}\right) \tag{2.33}$$

$$a = k_2^2 + k_1^2 + 4m^2, \qquad b = 2k_1 k_2.$$

Since $B(k_1)$ is a positive continuous function of k_1, we must only investigate its behaviour for $k_1 \to 0$ and $k_1 \to \infty$. For $k_1 \to 0$ we find from (2.33) $B(0) = 0$, and expanding the last factor in the integrand

$$\lim_{k_1 \to 0} \frac{1}{k_1^2} B(k_1) = 4\pi \int_0^\infty dk_2 \int_{-1}^{+1} dz \frac{k_2^2}{(k_2^2 + 4m^2)^{1+\varepsilon/2}} \times \left(\frac{1}{k_2^2 + 4m^2} - \frac{k_2^2 z^2}{(k_2^2 + 4m^2)^2}\right) = \text{const.} \tag{2.34}$$

The positive constant is bounded below uniformly in $\varepsilon > 0$.

For $k_1 \to \infty$ we transform the variables in (2.33) as follows

$$k_2' = \frac{k_2}{k_1}, \qquad a' = \frac{a}{k_1^2} = k_2'^2 + 1 + 4\frac{m^2}{k_1^2}, \qquad b' = \frac{b}{k_1^2} = 2k_2', \tag{2.35}$$

then

$$B(k_1) = k_1^{1-\varepsilon} 2\pi \int_0^\infty dk_2' \int_{-1}^{+1} dz \frac{k_2'^2}{(\frac{1}{2}\sqrt{a' + b'z} + \frac{1}{2}\sqrt{a' - b'z})^{2+\varepsilon}} \times \left(1 - \frac{k_2'^2 - 1 + 4m^2/k_1^2}{\sqrt{a'^2 - b'^2 z^2}}\right) \tag{2.36}$$

and

$$\lim_{k_1 \to \infty} \frac{1}{k_1^{1-\varepsilon}} B(k_1) = \text{const.}, \tag{2.37}$$

where again the constant is uniformly bounded below in ε. Equations (2.34) and (2.37) imply (2.31).

Applying similar arguments directly to the first condition (2.25), we find the following stronger ultraviolet condition

Theorem 3. *If $\hat{V}$ is a regular electrostatic potential, it satisfies*

$$\int_{n_1 N \leq p \leq n_2 N} d^3p\, p |\hat{V}(\mathbf{p})|^2 < C \tag{2.38}$$

for all $N \geq 0$ and fixed $n_2 > n_1 > 0$ with some constant C independent of N.

Proof. As in the proof of the foregoing theorem we write the condition (2.25) in terms of the relative coordinates $\mathbf{k}_1$, $\mathbf{k}_2$ (2.16)

$$J = \int d^3k_1\, B(k_1, \eta)|\hat{V}(\mathbf{k}_1)|^2 \leq C \tag{2.39}$$

where

$$B(k_1, \eta) = \eta^2 32\pi \int_0^\infty dk_2 \int_{-1}^{+1} dz \frac{k_2^2}{(a + 4\eta^2)^2 - b^2z^2}\left(1 - \frac{k_2^2 - k_1^2 + 4m^2}{\sqrt{a^2 - b^2z^2}}\right). \tag{2.40}$$

Here we introduce the scaled variables (2.35) again, scaling the parameter η also

$$\eta' = \frac{\eta}{k_1}. \tag{2.41}$$

Then

$$\lim_{k_1 \to \infty} \frac{1}{k_1} B(k_1, \eta) = \eta'^2 32\pi \int_0^\infty dk_2' \int_{-1}^{+1} dz \frac{k_2'^2}{(k_2'^2 + 1 + 4\eta'^2)^2 - 4k_2'^2z^2}$$
$$\times \left(1 - \frac{k_2'^2 - 1}{\sqrt{(k_2'^2 + 1)^2 - 4k_2'^2z^2}}\right) \stackrel{\text{def}}{=} f(\eta'). \tag{2.42}$$

The positive continuous function $f(\eta')$ vanishes only for $\eta' = 0$ and $\eta' \to \infty$. Choosing $0 < \eta_1' < \eta_2'$ fixed, we therefore have

$$f(\eta') \geq d > 0 \quad \text{for } \eta_1' \leq \eta' \leq \eta_2'. \tag{2.43}$$

For large enough $\eta = \eta_0$, it then follows from (2.39)

$$C \geq J \geq d \int_{\eta_0/\eta'_2 \leq k_1 \leq \eta_0/\eta'_1} d^3k_1 k_1 |\hat{V}(\mathbf{k}_1)|^2, \tag{2.44}$$

which proves (2.38).

This theorem is stronger than Theorem 2. For example, the potential $\hat{V}(p) = \sqrt{\log p}/p^2$ is not regular according to Theorem 3, while Theorem 2 does not exclude it. It is a pity that Theorem 3 is still not strong enough to decide the non-regularity of the Coulomb potential $\hat{V}(p) = 1/p^2$, which is expected from $V \notin A_0$ (1.3).

3. A sufficient condition

According to Theorem 1, it is sufficient to consider electrostatic potentials only in the following. Here we will prove the Hilbert–Schmidt property of

$$Q = P_+^0 - P_+ \tag{3.1}$$

for potentials in the class A_ε.

Theorem 4. *If*

$$\int d^3p \frac{|\mathbf{p}|^2}{1 + |\mathbf{p}|^{1-\varepsilon}} |\hat{V}(\mathbf{p})|^2 < \infty \tag{3.2}$$

for some $\varepsilon > 0$ *and* 0 *is not in the spectrum of H, then Q is a Hilbert–Schmidt operator.*

For the sake of clarity, we shall divide the proof in a number of steps. Our main tool is the well-known integral representation for the spectral projections in terms of the resolvents [3]

$$P_+ = \frac{1}{2} + \frac{1}{2\pi}\int_{-\infty}^{+\infty} R(i\eta)\, d\eta \tag{3.3}$$

which by means of (2.9) leads to

$$Q = P_+^0 - P_+ = \frac{1}{2\pi}\int_{-\infty}^{+\infty} R_0(i\eta) V R(i\eta)\, d\eta. \tag{3.4}$$

It is essential in our proof that this equation can be written in a more transparent form:

Lemma 1.

$$Q = \frac{1}{2\pi}\int_{-\infty}^{+\infty} R_0(i\eta)[P_+^0 V P_- + P_-^0 V P_+] R(i\eta)\, d\eta. \tag{3.5}$$

Proof. Since

$$P_-^0(P_+^0 - P_+)P_- = 0$$

we get from (3.4)

$$0 = P_-^0 Q P_- = P_-^0 \frac{1}{2\pi}\int_{-\infty}^{+\infty} R_0(i\eta) V R(i\eta)\, d\eta P_-$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} R_0(i\eta) P_-^0 V P_- R(i\eta)\, d\eta$$

where P_-^0, P_- can be put under the integral because they are bounded operators. Since

$$P_+^0 - P_+ = -(P_-^0 - P_-)$$

we also have

$$\int_{-\infty}^{+\infty} R_0(i\eta) P_+^0 V P_+ R(i\eta)\, d\eta = 0,$$

which proves (3.5).

Lemma 2. *Let* $Q = P_+^0 - P_+$. *Then*

$$Q = \frac{1}{2\pi}\int_{-\infty}^{+\infty} R_0(i\eta)(P_+^0 V P_-^0 + P_-^0 V P_+^0) R(i\eta)\, d\eta$$
$$+ \frac{1}{2\pi}(P_+^0 - P_-^0)\int_{-\infty}^{+\infty} R_0(i\eta) V Q R(i\eta)\, d\eta. \tag{3.6}$$

Proof. This relation follows from Lemma 1 (3.5) by adding and subtracting the first term on the right-hand side of (3.6).

The main idea of the proof of Theorem 2 is the solution of an equation like (3.6) for Q. This is done in the following

Lemma 3. *If*

$$Q = Y + \int_{-\infty}^{+\infty} A(\eta)QB(\eta)\, d\eta, \tag{3.7}$$

where Y is a Hilbert–Schmidt operator and

$$\int_{-\infty}^{+\infty} \|A(\eta)\|\;\|B(\eta)\|\, d\eta < 1, \tag{3.8}$$

then Q is also a Hilbert–Schmidt operator.

Proof. Under the condition (3.8) the equation (3.7) can be solved by iteration:

$$Q = Y + \sum_{j=1}^{\infty} \int d\eta_1 \ldots \int d\eta_j\, A(\eta_1)\ldots A(\eta_j)YB(\eta_j)\ldots B(\eta_1)$$

with

$$\|Q\|_{\text{H.S.}} \le \|Y\|_{\text{H.S.}} + \sum_{j=1}^{\infty} \|Y\|_{\text{H.S.}} \left(\int \|A(\eta)\|\;\|B(\eta)\|\, d\eta \right)^j < \infty.$$

Next, we need some technical results.

Lemma 4. *Let $V \in A_\varepsilon$. Then for any $\delta < \varepsilon$, V is $|H_0|^{1-\delta}$-compact, i.e. $|H_0|^{\delta-1}V$ is compact (where this operator is defined by extending it to the whole Hilbert space by continuity).*

Proof. First let us consider potentials $W(\mathbf{x}) \in C_0^\infty$. The operator

$$(W|H_0|^{\delta-1})(W|H_0|^{\delta-1})^+ = W|H_0|^{2\delta-2}W \tag{3.9}$$

is an integral operator with the kernel

$$K(\mathbf{x}, \mathbf{y}) = W(\mathbf{x})G_\alpha(\mathbf{x} - \mathbf{y})W(\mathbf{y}) \tag{3.10}$$

where G_α is the Bessel potential [4] for $\alpha = 2 - 2\delta$. Since $G_\alpha(\mathbf{x})$ is square-integrable at $\mathbf{x} = 0$ for $\alpha > \frac{3}{2}$ [5], it follows that (3.9) is a Hilbert–Schmidt operator for $\delta < \frac{1}{4}$. Hence, $|H_0|^{\delta-1}W$ is compact.

Next let $W(\mathbf{x}) \in L^p$ with $3/(1 - \delta) < p < \infty$. Then $W(\mathbf{x})$ can be approximated by C_0^∞-functions in the L^p norm

$$W = W_{1,n} + W_{2,n} \qquad W_{1,n} \in C_0^\infty, \qquad \|W_{2,n}\|_p \le \frac{1}{n}. \tag{3.11}$$

The Hölder and Hausdorff–Young inequalities imply

$$\|\,|H_0|^{\delta-1}W_{2,n}\| \le \text{const} \left\| \frac{1}{E(\mathbf{k})^{1-\delta}} \right\|_p \|W_{2,n}\|_p = \frac{\text{const}}{n}, \tag{3.12}$$

i.e. $|H_0|^{\delta-1}W$ is the norm limit of a sequence of compact operators and therefore compact.

Finally, let $V \in A_\varepsilon$. Then we decompose $\hat{V}$ in p-space

$$\hat{V}(\mathbf{k}) = \hat{V}_1(\mathbf{k}) + \hat{V}_2(\mathbf{k})$$

$$\hat{V}_1(\mathbf{k}) = \hat{V}(\mathbf{k})\chi_1(k), \qquad \hat{V}_2(\mathbf{k}) = \hat{V}(\mathbf{k})(1 - \chi_1(k))$$

$$\chi_1(k) = \begin{cases} 1 & |\mathbf{k}| \le 1 \\ 0 & |\mathbf{k}| > 1. \end{cases} \tag{3.13}$$

By Hölders inequality, we get

$$\| \hat{V}_1 \|_q \le \| \, |E(\mathbf{k})| \hat{V}_1 \|_2 \left\| \frac{1}{|E(\mathbf{k})|} \right\|_{2q/(2-q)} < \infty$$

if $\frac{6}{5} < q < 2$, which implies $V_1(\mathbf{x}) \in L^P$, $p \in [2, 6)$ by the Hausdorff–Young inequality. For $\hat{V}_2$ we have

$$\int (1 + |\mathbf{k}|^2)^{(1+\varepsilon)/2} |\hat{V}_2(\mathbf{k})|^2 < \infty,$$

because

$$(1 + |k|^2)^{(1+\varepsilon)/2} \le \text{const} \frac{k^2}{1 + k^{1-\varepsilon}} \quad \text{for} \quad |\mathbf{k}| > 1.$$

Then, by the convolution theorem

$$V_2(\mathbf{x}) = (G_{(1+\varepsilon)/2} * g)(\mathbf{x}) \tag{3.14}$$

with $g \in L^2$. Since $G_\alpha(\mathbf{x}) \in L^q$, $q \in [1, 3/(3 - \alpha))$, [5], the Young inequality implies $V_2(\mathbf{x}) \in L^P$, $p \in [2, 6/(2 - \varepsilon))$. Summing up, it follows that $|H_0|^{\delta - 1} V$ is compact for $2\delta < \varepsilon$, which completes the proof of Lemma 4.

Lemma 5. *Let $\delta > 0$. Then there exists $K_\delta < \infty$ such that*

$$\| R_0(i\eta) |H_0|^{1-\delta} \| \le \frac{K_\delta}{1 + \eta^\delta}. \tag{3.15}$$

Proof. The proof is a simple application of the functional calculus for self-adjoint operators.

Lemma 6. *Let $V \in A_\varepsilon$, $\varepsilon > 0$. Then*

$$Y_1 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} R_0(i\eta)(P_+^0 V P_-^0 + P_-^0 V P_+^0) R(i\eta) \, d\eta \tag{3.16}$$

is a Hilbert–Schmidt operator.

Proof. Separating the first order contribution in V, we can write Y_1 as follows

$$Y_1 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} R_0(i\eta) V R_0(i\eta) \, d\eta$$

$$- \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\eta R_0(i\eta)(P_+^0 V P_-^0 + P_-^0 V P_+^0) R_0(i\eta) V R_0(i\eta)(1 + V R_0(i\eta))^{-1}$$

$$\stackrel{\text{def}}{=} Y_1^{(1)} + Y_1^{(2)}. \tag{3.17}$$

It has been shown in [1] that $Y_1^{(1)}$ is a Hilbert–Schmidt operator. In particular, the Hilbert–Schmidt norm is given by ([1] (4.13))

$$\|Y_1^{(1)}\|_{\text{H.S.}}^2 = 2(2\pi)^3 \int d^3p \int d^3q \frac{|\hat{V}(\mathbf{p}-\mathbf{q})|^2}{(E_p + E_q)^2}\left(1 - \frac{\mathbf{p}\cdot\mathbf{q} + m^2}{E_p E_q}\right) < \infty \tag{3.18}$$

which is finite for $V \in A_0 \subset A_\varepsilon$ (1.2). Now

$$\|Y_1^{(2)}\|_{\text{H.S.}}^2 \leq \frac{1}{2\pi}\int d\eta \|R_0(i\eta)(P_+^0 V P_-^0 + P_-^0 V P_+^0)R_0(i\eta)\|_{\text{H.S}}$$

$$\times \|V|H_0|^{\delta-1}\| \; \||H_0|^{1-\delta}R_0(i\eta)\| \; \|(1 + VR_0(i\eta))^{-1}\|. \tag{3.19}$$

Since

$$\|R_0(i\eta)(P_+^0 V P_-^0 + P_-^0 V P_+^0)R_0(i\eta)\|_{\text{H.S.}}^2$$
$$= 2\int d^3p \int d^3q \frac{|\hat{V}(\mathbf{p}-\mathbf{q})|^2}{(E_p^2 + \eta^2)(E_q^2 + \eta^2)}\left(1 - \frac{\mathbf{p}\cdot\mathbf{q} + m^2}{E_p E_q}\right), \tag{3.20}$$

we get by means of the simple estimate

$$(E_p^2 + \eta^2)(E_q^2 + \eta^2) \geq \tfrac{1}{8}(2\eta^2 + m^2)(E_p + E_q)^2 \tag{3.21}$$

for the Hilbert–Schmidt norm of $Y_1^{(2)}$

$$\|Y_1^{(2)}\|_{\text{H.S.}}^2 \leq \frac{2}{\pi}\left\{\int d^3p \int d^3q \frac{|\hat{V}(\mathbf{p}-\mathbf{q})|^2}{(E_p + E_q)^2}\left(1 - \frac{\mathbf{p}\cdot\mathbf{q} + m^2}{E_p E_q}\right)\right\}^{1/2}$$

$$\times \left\{\int_{-\infty}^{+\infty} \frac{d\eta}{(2\eta^2 + m^2)^{1/2}} \| |H_0|^{1-\delta}R_0(i\eta)\|\right\} \|V|H_0|^{\delta-1}\| \sup_{\eta\in\mathbb{R}} \|(1 + VR_0(i\eta))^{-1}\| < \infty.$$

This is finite because of (3.18) and Lemma 4 and 5, which completes the proof of Lemma 6.

Proof of Theorem 4. Let $\gamma > 0$. Since $|H_0|^{\delta-1}V$ is compact, it can be written as

$$|H_0|^{\delta-1}V = M + E \tag{3.22}$$

where M is a Hilbert–Schmidt operator (actually M is finite dimensional) and

$$\|E\| \leq \gamma. \tag{3.23}$$

Then we can write (3.6) as follows

$$Q = \frac{1}{2\pi}\int_{-\infty}^{+\infty} R_0(i\eta)(P_+^0 V P_-^0 + P_-^0 V P_+^0)R(i\eta)\, d\eta$$
$$+ \frac{1}{2\pi}(P_+^0 - P_-^0)\int_{-\infty}^{+\infty} R_0(i\eta)|H_0|^{1-\delta}MQR(i\eta)\, d\eta$$
$$+ \frac{1}{2\pi}(P_+^0 - P_-^0)\int_{-\infty}^{+\infty} R_0(i\eta)|H_0|^{1-\delta}EQR(i\eta)\, d\eta$$

$$\stackrel{\text{def}}{=} Y + \int_{-\infty}^{+\infty} A(\eta) Q B(\eta)\, d\eta \tag{3.24}$$

with

$$A(\eta) = \frac{1}{2\pi}(P_+^0 - P_-^0) R_0(i\eta) |H_0|^{1-\delta} E$$
$$B(\eta) = R(i\eta). \tag{3.25}$$

For γ sufficiently small, we clearly have

$$\int_{-\infty}^{+\infty} \|A(\eta)\| \, \|B(\eta)\| \, d\eta < 1.$$

Since

$$\left\| \frac{1}{2\pi}(P_+^0 - P_-^0) \int R_0(i\eta)|H_0|^{1-\delta} M Q R(i\eta)\, d\eta \right\|_{\text{H.S}}$$
$$\leq \|M\|_{\text{H.S.}} \frac{2}{\pi} \int \|R_0(i\eta)|H_0|^{1-\delta}\| \, \|R(i\eta)\| \, d\eta < \infty,$$

Y in (3.24) is a Hilbert–Schmidt operator. Then it follows from Lemma 3 that Q is also a Hilbert–Schmidt operator. This completes the proof of the theorem.

Acknowledgement

One of us (G.N.) thanks the Swiss National Science Foundation for financial support during his visit at the Institut für Theoretische Physik der Universität Zürich where this work has been done.

REFERENCES

[1] M. KLAUS and G. SCHARF, *The regular external field problem in Q.E.D.*, Helv. Phys. Acta, *50*, 779 (1977).
[2] M. KLAUS and G. SCHARF, *Vacuum polarization in Fock space*, Helv. Phys. Acta, *50*, 803 (1977).
[3] T. KATO, *Perturbation Theory for Linear Operators*, 2nd Edn., (Berlin, Heidelberg, New York), 1976, p. 359.
[4] E. M. STEIN, *Singular Integrals and Differentiability Properties of Functions* (Princeton 1970), p. 130.
[5] E. M. STEIN, *Singular Integrals and Differentiability Properties of Functions* (Princeton 1970), p. 132.