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Autor(en): **Trucco, Ernesto**

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On the Stability of Heavy Nuclei in the Strong-Coupling Limit of the Pseudovector Meson Theory

by **Ernesto Trucco** (University of Chicago, Chicago, Illinois).

(9. II. 53.)

Abstract: Following closely a method introduced by WENTZEL, the stability of a hypothetical heavy nucleus with *pseudovector* meson forces acting between the individual nucleons is investigated on the assumption of *strong coupling*. This is done by computing the total nuclear energy as a function of the nuclear volume, the calculations being restricted to the case of sufficiently small nuclear radii. The resulting energy vs. volume curves are very similar to those found by WENTZEL for the case of scalar interaction: they show no minimum which would correspond to a position of stable binding; also the nuclear forces do not exhibit any saturation effects. Our model, therefore, does not adequately represent the basic properties of an actual nucleus.

1. Introduction.

In treating the *strong-coupling* approximation of meson field theories and its application to the problem of nuclear forces, WENTZEL[1]) first pointed out that a very stringent test for the usefulness of each type of meson field can be obtained by investigating the behavior of *heavy nuclei*. His calculations were based on the scalar theory; the method consisted in using the *Thomas-Fermi statistical* approach with a modification appropriate to take into account the *nuclear internal degrees of freedom* which characteristically appear in the case of strong coupling.

F. COESTER[2]) treated on similar lines the neutral and symmetrical pseudoscalar as well as the symmetrical vector theories*), showing that in these cases a stable heavy nucleus can exist, provided some parameters entering the theory are suitably chosen. In the following we shall assume the knowledge of both reff[1])[2]) and only sketch very briefly some of their most important features.

It is the purpose of this paper to apply the same method of analysis to the *symmetrical pseudovector theory*, the details of which have recently been worked out by RÜDENBERG[3]).

*) Including also the MØLLER-ROSENFELD mixture of symmetrical vector and pseudoscalar fields.

The results will turn out to be very similar to those found by WENTZEL for the scalar theory, leading to the conclusion that a heavy nucleus cannot be stable if pseudovector meson forces with strong coupling are assumed to act between the individual nucleons and that this type of interaction must therefore be discarded.

At the end of this paper we shall also say a few words on the validity of the conclusions reached by means of the THOMAS-FERMI approximation and on some objections that might be raised against it.

2. Brief survey of strong-coupling vector and pseudovector theory. The statistical model and the semiclassical approximation.

RÜDENBERG's paper follows closely WENTZEL's work on the symmetrical vector theory[4]). In both cases the first step is to set up the Hamiltonian which is composed of two parts, H_0 and H_1, the former referring to the free meson field and the latter describing the interaction between mesons and nucleons. Next, the one-nucleon problem is solved and finally an expression for the force between two particles is derived in the limiting case of fixed nucleons (static approximation). The whole theory is essentially non-relativistic, since it requires the introduction of a "cut-off", or, in other words, a finite size for the nucleons.

The most general form of H_1 compatible with invariance requirements consists of the sum of two terms, each of them affected by a coupling constant, f and g respectively. However, RÜDENBERG (in contrast to WENTZEL) confined himself to the cases in which only one of these constants (either f or g) is $\neq 0$, and we shall do likewise.

We must refer to the above-mentioned papers for all details of the rather cumbersome calculations, in particular as regards the conditions that limit the applicability of the theory, i. e. the inequalities that must be satisfied by f and g in order that the coupling may be termed strong. It will suffice here to say that a fundamental and characteristic difference between the strong-coupling and the usual weak-coupling approximation manifests itself in the treatment of the one-nucleon problem. By a series of canonical transformations new dynamical variables are introduced to describe the single nucleon. The physical interpretation of this procedure is that the originally "naked" nucleon attaches some mesons so strongly to itself as to form a new unit (the "complex nucleon") which is supposed to coincide with the observed or observable nucleon. In the

symmetrical vector and pseudovector theory the coordinates of the complex nucleon are:

$$\left.\begin{aligned} &q=(q^1, q^2, \ldots\ldots q^6) \text{ with} \\ &q^1=x_1\,, \qquad q^2=x_2\,, \qquad q^3=x_3\,, \\ &q^4=\cos\Theta=u\,, \qquad q^5=\Phi, \qquad q^6=\Psi\,. \end{aligned}\right\} \tag{1}$$

$$(-1\leqslant u\leqslant 1; \quad 0\leqslant\Phi<2\pi; \quad 0\leqslant\Psi<2\pi)\,. \tag{2}$$

q^1, q^2 and q^3 determine the position in space*); in a modellistic picture, in which the complex nucleon is represented as a *spherical top*, q^4, q^5 and q^6 are interpreted as Eulerian angles; they refer to the internal degrees of freedom, i. e. charge and angular momentum of the nucleon.

As a consequence of this situation the strong-coupling theory also predicts the existence of stationary states of the complex nucleon with higher charge and spin values (so-called isobar states), giving rise to an "energy of internal excitation" or isobar energy. In a classical approximation, i. e. neglecting the commutators $[p, q]$, this energy is given by:

$$H_{\mathrm{I}}=\varepsilon\left\{(1-u^2)\,p_u^2+\frac{1}{(1-u^2)}\,(p_\Phi^2+p_\Psi^2+2\,u\,p_\Psi p_\Phi)\right\} \tag{3}$$

which, in its dependence on the q^4, q^5, q^6 and their conjugate moments, corresponds exactly to the kinetic energy of a spherical top and is thus essentially positive. The quantity $\varepsilon\,(>0)$ may be assumed to be a constant in first approximation; its magnitude depends on the coupling parameters f and g. In a quantized theory the internal energy H_{I} is treated as an operator and its eigenstates or "stationary rotational states" determine the isobar energy levels of the complex nucleon. The two lowest isobar levels correspond to the usual neutron and proton states, and the internal energy of the next excited state is of the order of magnitude ε.

If we denote by E_{kin} the sum of kinetic and isobar energy, we may write:

$$E_{\mathrm{kin}}=\frac{1}{2\,M}\sum_{i=1}^{6}\sum_{k=1}^{6}g^{ik}\,p_i\,p_k\equiv\frac{1}{2\,M}\,|\,p\,|^2, \tag{4}$$

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*) We shall write x for $(q^1, q^2, q^3)\equiv(x_1, x_2, x_3)$.

where M is the mass of the nucleon and the "metric tensor" g^{ik} is given by:

$$\left.\begin{aligned} g^{ik} &= \delta^{ik} \text{ if } i \text{ or } k \leqslant 3, \\ g^{44} &= 2M\varepsilon(1-u^2) \\ g^{55} &= g^{66} = \frac{2M\varepsilon}{(1-u^2)} \\ g^{56} &= g^{65} = \frac{2M\varepsilon u}{(1-u^2)} \\ g^{45} &= g^{46} = g^{54} = g^{64} = 0. \end{aligned}\right\} \tag{5}$$

Finally, we need the expression for the nuclear interaction potential $V(q', q'')$. In the static approximation this is given by (cfr.[3]):

a) symmetrical pseudovector theory with $f = 0$:

$$V(q', q'') = -\left(\frac{g}{\mu}\right)^2 \sum_{\varrho=1}^{3} \sum_{i,j=1}^{3} s_{j\varrho}(\omega'')\, s_{i\varrho}(\omega') \frac{\partial^2}{\partial x_i' \partial x_j'} J(|x'-x''|)$$

b) symmetrical pseudovector theory with g = 0:

$$V(q', q'') = -\left(\frac{f}{\mu}\right)^2 \sum_{\varrho=1}^{3} \sum_{i,j=1}^{3} s_{j\varrho}(\omega'')\, s_{i\varrho}(\omega') \left[\delta_{ij}\mu^2 - \frac{\partial^2}{\partial x_i' \partial x_j'}\right] \times \times J(|x'-x''|). \tag{6}$$

Here:

$$J(|x'-x''|) = \frac{e^{-\mu|x'-x''|}}{4\pi|x'-x''|}, \quad \omega \equiv (u, \Phi, \Psi), \tag{7}$$

the meaning of the $s_{ij}(\omega)$ will be given later (see (10)) and μ is the meson mass*).

As in WENTZEL's and COESTER's papers the leading feature of the present investigation will be to compute the total nuclear energy as a function of the nuclear volume. To that end we use the statistical approach. Let us first write down the expression for the potential energy. According to Hartree and Fock this is given by:

$$E_{\text{pot}} = \frac{1}{2}\int\int dq'\, dq'' \left[\varrho(q')\,\varrho(q'') - \varrho(q', q'')\,\varrho(q'', q')\right] V(q', q'') \tag{8}$$

where $\varrho(q', q'') \equiv (q'|\varrho|q'')$ is Dirac's mixed density and $\varrho(q) \equiv \varrho(q, q)$.

The coordinates (1) have to be inserted for q', q'' and the potentials (6) for V. However, in treating a nucleus composed of many particles, we may replace the non-central part of the potential by

*) Our units are such that $\hbar = c = 1$.

its average over all spatial directions (as was done by COESTER*)) and use the following simplified expressions for V:

$$\left.\begin{aligned} &\text{a)} \quad V(q',q'') = -\frac{g^2}{3}\cdot J(|x'-x''|)\sum_{k=1}^{9} s_k(\omega')s_k(\omega'') \\ &\text{b)} \quad V(q',q'') = -\frac{2}{3}f^2 J(|x'-x''|)\sum_{k=1}^{9} s_k(\omega')\,s_k(\omega'')\,. \end{aligned}\right\} \tag{9}$$

In these formulae the double subscripts $i\varrho$ and $j\varrho$ have been replaced by the single index k now running from 1 to 9. The quantities $s_k(\omega)$ are essentially the coefficients of an orthogonal transformation (rotation) in 3-dimensional space expressed as functions of the Eulerian angles Θ, Φ and Ψ. We have:

$$\left.\begin{aligned} s_1(\omega) &= \sin\Phi\sin\Psi + \cos\Phi\cos\Theta\cos\Psi \\ s_2(\omega) &= -\sin\Phi\cos\Psi + \cos\Phi\cos\Theta\sin\Psi \qquad s_3(\omega) = \cos\Phi\sin\Theta \\ s_4(\omega) &= -\cos\Phi\sin\Psi + \sin\Phi\cos\Theta\cos\Psi \\ s_5(\omega) &= \cos\Phi\cos\Psi + \sin\Phi\cos\Theta\sin\Psi \qquad s_6(\omega) = \sin\Phi\sin\Theta \\ s_7(\omega) &= -\sin\Theta\cos\Psi \\ s_8(\omega) &= -\sin\Theta\sin\Psi \qquad\qquad\qquad s_9(\omega) = \cos\Theta\,. \end{aligned}\right\} \tag{10}$$

In writing the potential energy we have neglected the Coulomb interaction; for the moment we shall furthermore retain only the first term of (8), i. e. the non-exchange part of the potential energy. We are then left with:

$$E_{\text{Pot}} = -\frac{1}{2}\lambda\int\int dq'\,dq''\,\varrho(q')\,\varrho(q'')\,J(|x'-x''|)\sum_k s_k(\omega')\,s_k(\omega'') \tag{11}$$

for the *pseudovector* theory. Here λ is a positive constant:

$$\lambda = (g^2/3) \quad \text{or} \quad (2\cdot f^2/3)\,. \tag{12}$$

We may mention at this point that a similar constant λ' appearing in the *vector* theory is *negative*. This difference in sign will turn out to be of great consequence for the stability of the nucleus (see sect. 3).

As discussed by COESTER, there are two limiting cases in which the spatial density $\varrho(q)$ appearing in (11) can be easily calculated.

*) I. e. we substitute:

$$\frac{\partial^2}{\partial x_i'\,\partial x_j'}J(|x'-x''|) \to \frac{1}{3}\delta_{ij}\nabla^2_{(')}J(|x'-x''|) \to \frac{1}{3}\delta_{ij}\mu^2 J(|x'-x''|)\,.$$

These are those of "large" and "small" nuclear radius. In the latter, to which we shall confine ourselves, we may assume many isobar states of the single nucleons to be highly excited*). For this reason, since the complex nucleon is a system possessing a classical analogue (namely a spherical top), we can use a semiclassical approximation, i. e. the approximation of large quantum numbers, just as in the theory of heavy atoms[5][6][7]. In this limit the commutators $[p, q]$ may be neglected, so that the use of formula (3) for H_I is justified.

The (p, q) space (phase space) is divided into cells of volume $h^6 = (2\pi)^6$; we introduce the 6-dimensional THOMAS-FERMI potential

$$U(q') = \int dq''\, \varrho(q'')\, V(q', q'') = -\lambda \int dq''\, \varrho(q'')\, J(|x' - x''|) \times \times \sum_k s_k(\omega')\, s_k(\omega'') \tag{13}$$

and construct the surface of constant energy

$$H(p, q) = \frac{1}{2M} \sum_{i,k} g^{ik} p_i p_k + U(q) = W\,. \tag{14}$$

The energy parameter W is determined by the condition that each of the cells, for which $H(p, q) < W$, shall be occupied by one particle, provided the x at which the cell is located lies inside the nuclear volume; all other cells shall be empty. As usual in this method, we thus assume the nucleus to be in the state of lowest energy compatible with the fact that the nucleons obey FERMI statistics. We have already mentioned that even in this case many isobar states will be excited for sufficiently small nuclear radius.

Thus we have for the density in phase space:

$$\varrho(p, q) = \begin{cases} \dfrac{1}{(2\pi)^6} & \text{if } H(p, q) < W \text{ and } x \text{ inside } v \\ 0 \text{ otherwise.} & (v = \text{nuclear volume.}) \end{cases} \tag{15}$$

The particle density is then given by:

$$\underbrace{\varrho(q) = \int dp\, \varrho(q, p)}_{0 < |p|^2 < 2M[W - U(q)]} \qquad \left(dp \equiv \prod_{i=1}^{6} dp_i\right), \tag{16}$$

*) See G. WENTZEL, l. c. [1]), § 3 for a more quantitative discussion of this point, based on the scalar theory. WENTZEL verifies a posteriori that, below a certain critical nuclear volume, the actual values of the variable p_ϑ are $\gg h$, so that the distribution in phase space may be replaced by a continuous function.

or:

$$\varrho(q) = \begin{cases} \dfrac{1}{(2\pi)^6} \cdot \dfrac{1}{\sqrt{\det(g^{ik})}} \cdot \pi^3 \dfrac{(2M[W-U(q)])^3}{6} = \\ = \left(\dfrac{2M}{\varepsilon}\right)^{3/2} \dfrac{1}{48 \cdot (2\pi)^3} [W-U(q)]^3 \\ \text{if } W-U(q) > 0, \text{ and } x \text{ inside } v, \\ 0 \text{ otherwise.} \end{cases} \tag{17}$$

Neglecting surface effects, we shall assume that neither $\varrho(q)$ nor $U(q)$ depend on x inside the nucleus:

$$\varrho(q) = \begin{cases} \varrho(\omega) \\ 0 \end{cases} \qquad U(q) = \begin{cases} U(\omega) & \text{inside the nucleus} \\ 0 & \text{outside the nucleus,} \end{cases} \tag{18}$$

and change (13) into:

$$U(\omega) = -A \int d\omega' \, \varrho(\omega') \sum_k s_k(\omega') \, s_k(\omega) \qquad (d\omega' \equiv d\Phi' \, d\Psi' \, du') , \tag{19}$$

where

$$A = \lambda \cdot 4\pi \int_0^\infty dr \, r^2 J(r) > 0 . \tag{20}$$

Putting:

$$C_k = -A \int d\omega' \, \varrho(\omega') \, s_k(\omega') , \tag{21}$$

(19) can be written:

$$U(\omega) = \sum_{k=1}^{9} C_k \, s_k(\omega) . \tag{22}$$

If N is the number of particles forming the nucleus ($N \gg 1$), we have:

$$N = \int dq \, \varrho(q) = v \int d\omega \, \varrho(\omega) . \tag{23}$$

The "total kinetic energy" (= kinetic + isobar energy) is obtained by adding the contributions from the single nucleons:

$$E_{\text{kin}} = \int\int dp \, dq \, \varrho(q, p) \cdot \frac{1}{2M} |p|^2 . \tag{24}$$

Again, the p-integral can be easily calculated; we find:

$$E_{\text{kin}} = \frac{6}{8} \int dq \, \varrho(q) \, [W - U(q)] ,$$

where $\varrho(q)$ is given by (17) and (18). With the help of (23) this becomes:

$$E_{\text{kin}} = \frac{3}{4} \left\{ NW - v \int d\omega \, \varrho(\omega) \, U(\omega) \right\} . \tag{25}$$

On the other hand, using (13) and (18) we can write for the potential energy (11):

$$E_{\text{Pot}} = \frac{1}{2}\, v \cdot \int d\omega\, \varrho(\omega)\, U(\omega)\,, \tag{26}$$

so that the total energy of the nucleus will be:

$$E = \frac{3}{4} NW - \frac{1}{4} v \int d\omega\, \varrho(\omega)\, U(\omega)\,. \tag{27}$$

Multiplying (21) by C_k, summing over k and using (22), we find:

$$-\frac{\sum_k C_k^2}{A} = \int d\omega\, \varrho(\omega)\, U(\omega)\,. \tag{28}$$

Hence (27) can be written:

$$E = \frac{3}{4} NW + \frac{1}{4A} \cdot v \cdot \sum_k C_k^2\,. \tag{29}$$

We now call Ω the whole domain of variability of the $\omega's$, defined by (2), and $\bar{\Omega}$ that part of Ω in which $W - U(\omega) > 0$. Also we put

$$y = (E/N) \quad \text{and} \quad x = (v/N)\ ^*)\,. \tag{30}$$

Collecting our formulae, we then have the following set of equations:

$$\varrho(\omega) = \left\{ \begin{array}{ll} B \cdot (W - U(\omega))^3 & \text{if} \quad W - U(\omega) > 0 \\ 0 & \text{otherwise} \end{array} \right\} \tag{I}$$

$$B > 0\,.$$

$$U(\omega) = \sum_{k=1}^{9} C_k\, s_k(\omega) \quad \big(s_k(\omega) \text{ given by (10)}\big)\,, \tag{II}$$

where:

$$C_k = -A \cdot B \int_{\bar{\Omega}} d\omega\, s_k(\omega) \left[W - \sum_{j=1}^{9} C_j s_j(\omega) \right]^3 \tag{III}$$

$$(k = 1, 2, \ldots\ldots 9)\,, \qquad \underline{A > 0}\,.$$

$$(N/v) = (1/x) = B \int_{\bar{\Omega}} d\omega \left[W - \sum_{k=1}^{9} C_k\, s_k(\omega) \right]^3\,. \tag{IV}$$

$$(E/N) = y = \frac{3}{4} W + \frac{x}{4A} \sum_{k=1}^{9} C_k^2\,. \tag{V}$$

*) Not to be confused with $x \equiv (q^1, q^2, q^3)$.

3. Coester's Theorem and the First Saturation Requirement.

COESTER's Theorem states that the quantity

$$\int_{\Omega} d\omega\, \varrho(\omega)\, U(\omega)$$

is $\leqslant 0$, and vanishes if and only if $\varrho(\omega)$ is a constant, in which case we have $U(\omega) \equiv 0$ (as follows from (19) and from the relations

$$\int_{\Omega} d\omega'\, s_k(\omega') = 0 \quad \text{for} \quad k = 1, 2, 3, \cdots 9)\,.$$

We therefore see from (28) that if the constant A (cfr. (20) and (12)) were < 0, our equations (I), (II), ... (V) could be satisfied only by the "trivial solution"

$$C_1 = C_2 = C_3 = \cdots\cdots = C_9 = 0\,. \tag{31}$$

This happens to be the case in the vector theory, and a similar situation holds for all types of meson fields treated by COESTER (notice that COESTER's A corresponds to $-(A/\lambda)$ in our notation).

Thus the sign of A is of primary importance in our statistical model of the nucleus. It was noticed by WENTZEL and COESTER that for $A < 0$ the nucleons tend to be uniformly distributed over the space of the angular variables: $\varrho(\omega) = \text{const.}$ and $\bar{\Omega} = \Omega$; whereas for $A > 0$ there takes place a peculiar "freezing" of the internal degrees of freedom: the domain $\bar{\Omega}$ within which $\varrho(\omega)$ is $\neq 0$ becomes smaller and smaller with decreasing nuclear radius. As shown by the calculations, this shrinking of $\bar{\Omega}$ is closely connected with the fact that the nuclear forces tend to loose their saturation character as the nuclear volume becomes very small.

In the usual weak-coupling approximation the isobar energy does not exist and the ordinary kinetic energy is roughly proportional to the number of nucleons N (see: ROSENFELD, l. c. §§ 9, 11, 12). Then a necessary condition in order that the nucleus should exhibit the observed saturation properties is that the non-exchange part of the potential energy be $\geqslant 0$. ROSENFELD calls this the first saturation requirement.

In our case, however, the situation is not quite so definite. It is true that (if the C's are not all zero) the potential energy (11) or (26) is < 0 by COESTER's theorem; yet, this effect might be compensated by the positive "total kinetic energy" which could be proportional to some power of N higher than the first for very dense nuclei (see also the end of sect. 4). In other words, the nuclear

motion (translational and internal) could counteract the strong attraction caused by the non-saturated nuclear forces and thus prevent a "collapsing" of the nucleus. It will turn out, however, that this is not the case, just as in the scalar theory.

4. Solution of the basic equations in two simple cases. Non-saturation character of the forces.

A and B being known constants, we now have to solve the equations (III) and (IV) for the ten unknown quantities $C_1, C_2, \ldots C_9$, W, and insert the values thus obtained into (V).

It seems hopeless, however, to attempt the solution of this mathematical problem in all its generality, as may already be seen by assuming $\bar{\Omega} = \Omega$. Then the integrals appearing in (III) and (IV) can be evaluated explicitely and we get ten equations of the third degree for W and the C's. Each real solution of this system gives a possible $y(x)$ curve, as long as

$$W - \sum_k C_k \cdot s_k(\omega)$$

remains > 0 for all values of the ω's.

To exemplify the procedure let us treat the very simple special case:

$$C_1 = C_2 = \cdots\cdots = C_8 = 0, \qquad C_9 = C \neq 0. \tag{32}$$

Let us also assume:

$$W \geqslant |C| > 0, \quad \text{i. e.} \quad \bar{\Omega} = \Omega. \tag{33}$$

The first eight of eqs. (III) are then automatically satisfied; the last one becomes:

$$C = -AB \int_0^{2\pi} \int_0^{2\pi} \int_{-1}^{1} d\Phi\, d\Psi\, du\, u\, [W - Cu]^3 = 8\pi^2 AB \left(CW^2 + \frac{1}{5} C^3\right)$$

or:

$$C^2 = (5/8\pi^2 AB) - 5 \cdot W^2. \tag{34}$$

Similarly we get from (IV):

$$1/x = 8\pi^2 B \cdot (W^3 + W \cdot C^2). \tag{35}$$

Inserting C^2 from (34) we then have a relation between W and x:

$$W^3 - (5/32 \cdot \pi^2 \cdot A \cdot B) \cdot W + (1/32 \cdot \pi^2 \cdot B) \cdot (1/x) = 0, \tag{36}$$

which permits us to calculate $W(x)$ and hence $C^2(x)$ from (34) and $y(x)$ from (V). Finally, by (34) and (36), the condition (33) amounts to the following restriction for the values of x:

$$4\pi\left(\frac{3A}{5}\right)^{3/2}\sqrt{B} \leqslant x \leqslant \pi(2A)^{3/2}\sqrt{B}\,. \tag{37}$$

Instead of carrying this discussion further, it is convenient to use a slightly different method, which will allow us to get rid of the assumption (33). Putting

$$t = (W/C)\,, \tag{38}$$

we try to express both x and y as functions of the parameter t. We then have to distinguish the following cases:

a) $C > 0,\ t > 1,\ \bar{\Omega} = \Omega.$

(34) and (35) become, with the help of (38):

$$C = \sqrt[+]{\frac{5}{8\pi^2 AB(5t^2+1)}} = C(t) > 0\,, \tag{39}$$

and:

$(1/B\cdot C^3\cdot x) = 8\pi^2(t^3+t)$, or, using (39):

$$x(t) = \pi\left(\frac{2A}{5}\right)^{3/2}\cdot\sqrt{B}\cdot\frac{(5t^2+1)^{3/2}}{t(t^2+1)}\,.\ ^{*)} \tag{40a}$$

From (V) we finally obtain:

$$y(t) = (3/4)\cdot t\cdot C(t) + (1/4\cdot A)\cdot x(t)\cdot[C(t)]^2 =$$

$$= \frac{1}{8\pi\sqrt{10\,AB}}\left\{\frac{15t^4+20t^2+1}{t(t^2+1)\sqrt{5t^2+1}}\right\}. \tag{40b}$$

b) $C > 0,\ -1 \leqslant t \leqslant 1.$

The domain $\bar{\Omega}$ is now determined by:

$$0 \leqslant \Phi < 2\pi\,,\qquad 0 \leqslant \Psi < 2\pi\,,\qquad -1 \leqslant u < t\,.^{**)} \tag{41}$$

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*) $x(1) = 4\pi\left(\frac{3A}{5}\right)^{3/2}\cdot\sqrt{B}$, $x(\infty) = \pi(2A)^{3/2}\sqrt{B}$ (see (37)).

**) Since $t = -1$ corresponds to $x = 0$ (cfr. (44a)), we see that the extension of $\bar{\Omega}$ tends to zero for $x \to 0$, as mentioned in sect. 3.

(IV) can be written:

$$(1/C^3 \cdot x) = 4\pi^2 B \int_{-1}^{+} du \cdot (t-u)^3 = \pi^2 \cdot B \cdot (1+t)^4, \tag{42}$$

(III) gives:

$$C(t) = \frac{1}{\pi (t+1)^2} \sqrt{\frac{5}{AB(4-t)}} > 0, \tag{43}$$

and in the same way as before we find:

$$x(t) = \pi \left(\frac{A}{5}\right)^{3/2} \sqrt{B}\, (4-t)^{3/2} (t+1)^2, \tag{44a}$$

$$y(t) = \frac{1}{2\pi \sqrt{5AB}} \frac{(7t+2)}{(t+1)^2 \sqrt{(4-t)}}. \tag{44b}$$

c) $C < 0$.

Introducing the parameter $t = -(W/C) = (W/|C|)$, we obtain the same equations as (40) and (44) (with $|C|$ instead of C) and therefore the same energy vs. nuclear volume curve.

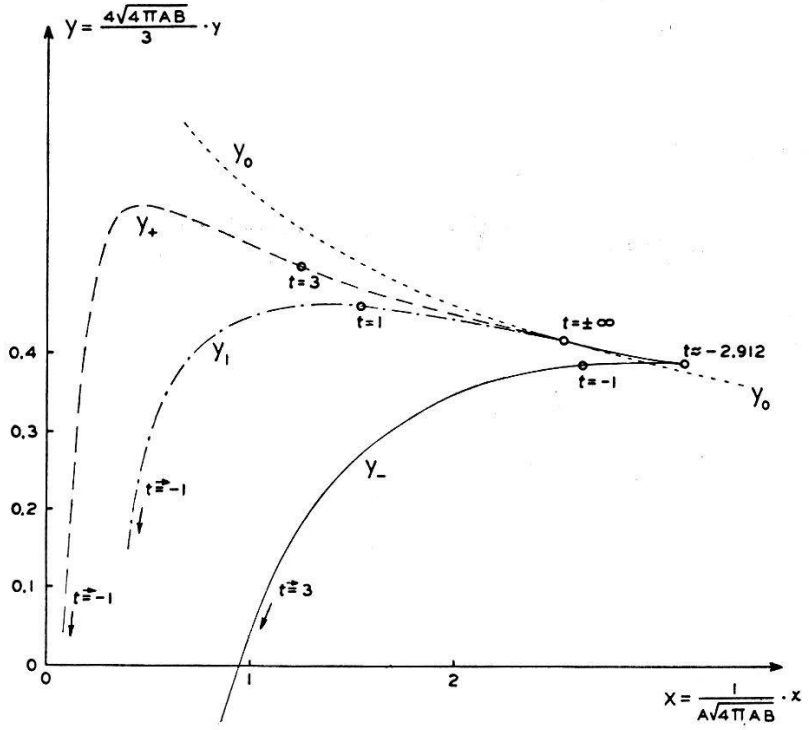

Fig. 1.

The curves y vs. x ($y = E/N$ $x = v/N$)

It is a matter of simple algebra to discuss the functions (40) and (44) and to construct the resulting curve $y(x)$ which is plotted in fig. 1 (denoted by $y_1(x)$ to distinguish it from other solutions).

Besides the "trivial solution" (31) which gives:

$$y_0(x) = (3/4) \cdot (8\pi^2 B)^{-1/3} \cdot x^{-1/3}, \tag{45}$$

another relatively simple case is obtained if we try to solve our equations on the assumption that:

$$\left.\begin{aligned} C_1 = C_5 = C_9 = C \neq 0 \\ C_2 = C_3 = C_4 = C_6 = C_7 = C_8 = 0\,. \end{aligned}\right\} \tag{46}$$

However, since the calculations pertaining to (46) are rather involved, they will not be reproduced here. The general procedure is the same as before; it is found that a separate discussion is necessary for:

$$\text{a) } C > 0, \qquad \text{and} \qquad \text{b) } C < 0.$$

We then put again $t = (W/C)$ and find

$$C(t),\ W(t) = t \cdot C(t),\ x(t) \text{ and } y(t).$$

It turns out that in case a) the parameter t can assume the values

$$-1 \leqslant t < \infty$$

and we have:

$$x(-1) = 0, \qquad y(-1) = -\infty,$$

$$\bar{\Omega} < \Omega \text{ for } -1 \leqslant t < 3, \qquad \bar{\Omega} = \Omega \text{ for } t \geqslant 3.$$

In case b), on the other hand, the possible values of t are:

$$-\infty < t \leqslant 3$$

with:

$$x(3) = 0, \qquad y(3) = -\infty,$$

$$\bar{\Omega} = \Omega \text{ for } -\infty < t \leqslant -1, \qquad \bar{\Omega} < \Omega \text{ for } -1 < t \leqslant 3.$$

It can also be verified by direct calculation that the equations (III) and (46) are compatible, i. e. that:

$$-AB \int_{\bar{\Omega}} d\omega\, s_k(\omega) \{t - [s_1(\omega) + s_5(\omega) + s_9(\omega)]\}^3 =$$

$$= \begin{cases} \dfrac{1}{[C(t)]^2} & \text{for } k = 1, 5, 9 \\ 0 & \text{for } k = 2, 3, 4, 6, 7, 8\,. \end{cases}$$

We denote the energy vs. nuclear volume curves resulting from (46) by $y_+(x)$ and $y_-(x)$, according to whether C is > 0 or < 0; they are also plotted in fig. 1, together with the trivial solution $y_0(x)$.

Closer inspection shows that the different sections of the various curves all join smoothly. In particular, at the point corresponding to $t = \pm\infty$ ($C = 0$) the four curves y_+, y_-, y_1 and y_0 have the same value and the same tangent.

As far as these calculations go, no stable nucleus can exist, since none of our curves $y(x)$ shows a minimum of the energy. The similarity between these results and those found by WENTZEL is evident, and in both cases the most striking feature of the curves is their asymptotic behavior in the vicinity of $x = 0$. Excluding the trivial solution ($y \sim x^{-1/3}$), we find that the product

$$x(t) \cdot y(t)$$

tends to a finite negative value as $t \to t_0$, where t_0 is such that $x(t_0) = 0$. This means that

$$E \sim -(N^2/v) \text{ as } v \to 0,$$

i. e. in this limit the nuclear forces behave like ordinary, non-saturated attractive forces, causing the nucleus to shrink to an infinitesimal volume.

It has already been noticed by WENTZEL (l. c.[1])) that neither the COULOMB energy nor the exchange energy, both of which we have hitherto neglected, can alter this result, and we shall not insist further on this point*).

The simple calculations based on (32) may also serve to illustrate the remark made at the end of sect. 3 on the influence of the kinetic energy. For $-1 \leqslant t \leqslant 1$ we have from (25), (28), (38), (43) and (44a):

$$(E_{\text{kin}}/N) = (3/4)\{t \cdot C(t) + (x(t)/A) \cdot [C(t)]^2\} =$$

$$= \frac{3}{\pi \sqrt{5\,AB}\,(t+1)\sqrt{(4-t)}} > 0\,.$$

Comparison with (44a) shows that

$$\frac{E_{\text{kin}}}{N} \sqrt{\frac{v}{N}}$$

*) In COESTER's case we have the opposite situation, since there the non-saturation part of the potential energy is $= 0$ and therefore the exchange energy plays an essential role in bringing about the stability of the nucleus. In fact, if both the COULOMB and the exchange energy were neglected, the $y(x)$ curve for $A < 0$ and small x would be given solely by our "trivial solution" which obviously does not correspond to any stable state of binding.

has a finite value for $t \to -1$, so that, in the limit of vanishing nuclear radius, the "total kinetic energy" increases only as $(N^{3/2}/\sqrt{v})$ in contrast to the potential energy which is proportional to $-(N^2/v)$ and thus becomes preponderant.

5. Concluding remarks.

The arguments presented in sects. 3 and 4 suggest strongly that, if $U(\omega) \not\equiv 0$, the energy will in all cases behave asymptotically as $E \sim -(N^2/v)$ for $v \to 0$; yet we did not succeed in finding a general mathematical proof for this fact. Fortunately, however, it is not necessary to deal with this problem, since, even if there existed a curve $y(x)$ with a minimum corresponding to a state of binding, the latter could only have a limited lifetime. The heavy nucleus, being a system with very many degrees of freedom, would soon perform

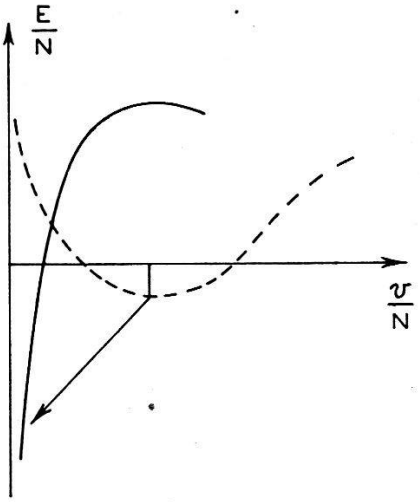

Fig. 2.

a transition to the energetically lower state of closest packing (fig. 2). The conclusions enunciated in sect. 1 therefore appear to be generally valid.

Our description of the nucleus is adequate only for sufficiently small values of $x = (v/N)$; within this range the approximations made, in particular the use of the THOMAS-FERMI method, seem quite reasonable. Perhaps the most serious omission concerns the surface effects. As well known, they are by no means unimportant[8]), but it is clear that they cannot prevent the shrinking of a nucleus composed of sufficiently many particles.

In his review of COESTER's work, ROSENFELD (l. c.[2])) states that the use of the statistical model is "far from reliable". This remark, however, would rather seem to apply to the quantitative conclusions

drawn by COESTER from his calculations than to the general qualitative features of the theory. Though FOCK's method is admittedly far less reliable in nuclear than in atomic physics, the only essential thing we have to prove is the inadequacy of the kinetic energy in preventing a "nuclear breakdown", since the first saturation requirement is not fulfilled. Remembering that the actual values of the energy are always lower than those resulting from FOCK's equation, we have no reason to doubt the validity of our conclusions.

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