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Loop Quantum Gravity and Black Hole Physics

By Carlo Rovelli[1]

Department of Physics and Astronomy,
University of Pittsburgh, Pittsburgh Pa 15260, USA

Abstract. I summarize the basic ideas and formalism of loop quantum gravity. I illustrate the results on the discrete aspects of quantum geometry and two applications of these results to black hole physics. In particular, I discuss in detail the derivation of the Bekenstein-Hawking formula for the entropy of a black hole from first principles.

1 Introduction

The lack of understanding of the quantum behavior of the gravitational field, and therefore the lack of understanding of quantum geometry, remains a major open problem in fundamental physics. There are tentative theories which are presently intensively investigated. For instance, there is perturbative string theory, and non-perturbative string theory, much studied in these days.[2] A less ambitious attempt to solve the problem is nonperturbative quantum gravity, or "loop quantum gravity".[3] This is the project of taking the conceptual novelties introduced in physics by general relativity very seriously, and trying to make sense of quantum general relativity (or any other high energy extension of general relativity) nonperturbatively.

The program is based on the hypothesis that perturbative approaches fail to describe Planck scale physics because at the Planck scale the separation between a background metric and a quantum field is not physically justified. At short scale, spacetime is not Minkoskian. A posteriori, loop quantum gravity supports the hypothesis, since the short structure of the

[1]E-mail: rovelli@pitt.edu
[2]For an overview of other current ideas on quantum geometry, see [1], [2], and [3]
[3]Ted Jacobson calls it "Loopy quantum gravity".

geometry turns out to be strongly non-Minkoskian in the nonperturbative theory [4]. Thus, *"Man shall not separate what Einstein put together"*: we should not separate the gravitational field from the metric.

If this idea is correct, then the quantum gravitational field cannot be described as a quantum field over a metric manifold, because there is no background field to provide the metric structure. Therefore we have to learn how to construct a quantum field theory living over a structure weaker than a metric manifold: namely over a differential manifold. The aim of nonperturbative quantum gravity is to understand what is quantum field theory (QFT) on differential manifolds (as opposed to QFT on metric spaces). In other words, we want a QFT which is formulated in a diffeomorphism invariant way, and therefore may incorporate the essential physical content of general relativity (GR), which –I am convinced– is encoded in its active diffemorphism invariance.

In such a context, most of the techniques of QFT that we like and love become useless. Therefore a QFT on a manifold turns out to have a structure profoundly different from usual QFT's. The manifold itself is "washed away" by diffeomorphism invariance, and therefore the excitations of the quantum field do not live "over a space"; they live "nowhere", since they *are* the space over which physics happens. In other words, QFT must undergo the same deep transformation that classical field theory had to undergo in the evolution from pre-general relativistic physics to general relativistic physics: In general relativistic physics, the "location" of physical objects and physical fields is not determined with respect to a preexisting space. Rather, physical quantities (which include the gravitational field) are only "located" with respect to each other [5]. The challenge of quantum gravity is to incorporate this relational notion of localization, introduced by general relativity, into QFT. As we shall see, the physical elementary excitations of the quantum gravitational field are described in loop quantum gravity by abstract objects (s-knots), which do not live "inside a given space". Rather, they *are* physical space, at the quantum level.

The resulting theory fails to satisfy even the simplest assumptions on which conventional local QFT is based. The theory is hard to analyze; it often contradicts our intuition and some of our accepted believes, developed in the context of local QFT's. This is a fact which unfortunately often complicates the communication between quantum gravity and other sectors of theoretical physics.

The idea of exploring quantum GR nonperturbatively is old. But during the last decade, the research program has developed intensely, prompted by two technical advances. One is the reformulation of classical general relativity due to Ashtekar [6], which has substantially simplified the formalism. The second is the introduction of the loop representation for quantum gravity [7, 8].[4] The loop representation is a technique for defining a nonperturbative quantum theory on a manifold. The idea is to replace creation and annihilation operators, which are the essential ingredients of conventional QFT and which make sense only if there is a background metric, with a different set of operators: the loop operators. The loop operators do not require a background metric to be defined. The theory defined by

[4]For a recent overview of canonical gravity, see [9]; for introductions to loop quantum gravity, see [10, 11, 12, 13, 14, 15, 16].

a representation of the creation and annihilation operator algebra is naturally in the Fock, or particle, basis. The theory defined by the loop operators is naturally in a basis, denoted the loop basis, or more precisely the spin network basis, which turns out to be particularly suitable for dealing with the quantum kinematics and dynamics of the theory, as well as for analising the nonperturbative aspects of quantum geometry.

The loop approach to nonperturbative quantum gravity has now developed in many directions. Here, I present a brief overview of the main ideas, techniques and results, and I focus on a particularly interesting result: the explicit computation of the spectrum of the area [17, 18, 19, 20].

A traditional problem in quantum gravity is the difficulty of testing ideas and results [2]. This is due not only to the lack of direct experimental or observational access to Planck scale physics; but also to the intrinsic difficulty of extracting results from the theory that could be tested indirectly. However, there is an area of theoretical physics that gives us indirect information on quantum gravity: black hole thermodynamics. The great power of thermodynamics to put constraints on theoretical constructions, and even provide precise quantitative indications on microscopic theories is well known: quantum mechanics itself was born to a large extent in order to satisfy thermodynamical consistency requirements (Planck's spectrum, solid state...). Now, black hole thermodynamics derives a surprising set of simple laws just from classical general relativity and quantum field theory in curved spacetime (for an introduction, see [21]). These laws have not been experimentally tested, but are very well motivated. However, they are thermodynamical "phenomenological" laws, and their derivation from first principles requires a quantum theory of gravity, and, at present, is lacking.

This state of affairs provides the ideal testing ground for loop quantum gravity. The study of the applications of loop quantum gravity to black hole thermodynamics has just begun. Here, I describe two of these applications. The first [22] is a discussion of the Bekenstein Mukhanov effect [23]. The second [24, 25] and more important application is a derivation of the Bekenstein-Hawking black hole entropy formula [26, 27] from first principles. In some parts of this lecture I will follow, and sometimes expand, references [16, 22, 24].

2 Overview of loop quantum gravity

Classical general relativity can be formulated in phase space form as follows [6, 28, 29]. We fix a three-dimensional manifold M and consider two real (smooth) $SO(3)$ fields $A_a^i(x)$ and $\tilde{E}_i^a(x)$ on M. We use $a, b, \ldots = 1, 2, 3$ for (abstract) spatial indices and $i, j, \ldots = 1, 2, 3$ for internal $SO(3)$ indices. We indicate coordinates on M with x. The relation between these fields and conventional metric gravitational variables is as follows: $\tilde{E}_i^a(x)$ is the (densitized) inverse triad, related to the three-dimensional metric $g_{ab}(x)$ of the constant-time surface by

$$g\, g^{ab} = \tilde{E}_i^a \tilde{E}_i^b, \tag{2.1}$$

where g is the determinant of g_{ab}; and

$$A_a^i(x) = \Gamma_a^i(x) + k_a^i(x); \tag{2.2}$$

Where $\Gamma_a^i(x)$ is the $SU(2)$ spin connection associated to the triad and $k_a^i(x)$ is the extrinsic curvature of the three surface (up to indices' position). Notice the absence of the i in (2.2), which yields the *real* Ashtekar connection.

The spinorial version of the Ashtekar variables is given in terms of the Pauli matrices $\sigma_i, i = 1, 2, 3$, or the $su(2)$ generators $\tau_i = -\frac{\mathrm{i}}{2}\,\sigma_i$, by

$$\tilde{E}^a(x) = -\mathrm{i}\,\tilde{E}_i^a(x)\,\sigma_i = 2\tilde{E}_i^a(x)\,\tau_i \tag{2.3}$$

$$A_a(x) = -\frac{\mathrm{i}}{2}\,A_a^i(x)\,\sigma_i = A_a^i(x)\,\tau_i\,. \tag{2.4}$$

$A_a(x)$ and $\tilde{E}^a(x)$ are 2×2 complex matrices.

The theory is invariant under local $SO(3)$ gauge, three-dimensional diffeomorphisms of the manifold on which the fields are defined, as well as under (coordinate) time translations generated by the Lorentzian Hamiltonian constraint. The full dynamical content of GR is captured by the three constraints that generate these gauge invariances [6]. The Lorentzian Hamiltonian constraint does not have a simple polynomial form if we use the real connection (2.2). For a while, this fact was considered an obstacle defining the quantum Hamiltonian constraint; therefore the complex version of the connection was mostly used. However, Thiemann has recently succeded in constructing a satisfactory Lorentzian quantum hamiltonian constraint [30] in spite of the non-polynomiality of the classical expression. This is the reason we use here the real connection. This choice has the advantage of greatly simplifying the "reality conditions" problem.

To construct the quantum theory, we have to promote the fields to operators on a Hilbert space. One possibility is to consider the positive and negative frequencies of A and E, and define a Fock representation. The definition of positive and negative frequencies requires a metric. Thus, one may consider an unperturbed background field around which expanding A and E, and use the unperturbed field as background metric. The problem is that the expansion becomes unsuitable precisely at the Planck scale, which is the scale we are interested in.

The loop representation is based on the choice of other quantities to be promoted as basic operators. These are: the trace of the holonomy of the Ashtekar connection, which is labeled by loops on the three manifold; and the higher order loop variables, obtained inserting the E field (in n distinct points, or "hands" of the loop variable) into the holonomy trace. More precisely, given a loop α and the points $s_1, s_2, \ldots, s_n \in \alpha$ we define:

$$\mathcal{T}[\alpha] = -\mathrm{Tr}[U_\alpha], \tag{2.5}$$

$$\mathcal{T}^a[\alpha](s) = -\mathrm{Tr}[U_\alpha(s,s)\tilde{E}^a(s)] \tag{2.6}$$

and, in general

$$\mathcal{T}^{a_1 a_2}[\alpha](s_1, s_2) = -\mathrm{Tr}[U_\alpha(s_1, s_2)\tilde{E}^{a_2}(s_2)U_\alpha(s_2, s_1)\tilde{E}^{a_1}(s_1)], \tag{2.7}$$

$$\mathcal{T}^{a_1 \ldots a_N}[\alpha](s_1 \ldots s_N) = -\mathrm{Tr}[U_\alpha(s_1, s_N)\tilde{E}^{a_N}(s_N)U_\alpha(s_N, s_{N-1}) \ldots \tilde{E}^{a_1}(s_1)]$$

where U is the parallel propagator of A_a along γ. (See [16] for more details.) These are the loop observables. They coordinatize the phase space and have a closed Poisson algebra. Thus, we may pick a unitary representation of this algebra as the definition of the kinematic of the quantum theory.

2.1 The Hilbert space

A representations of the loop algebra can be defined as follows (The first introduction of the loop representation, in the context of Yang Mills theory, is in [31]). We consider the free algebra $\mathcal{A}^f[\mathcal{L}]$ over the set of the loops in the three manifold, namely the set of objects Φ which are (finite) formal linear combinations of formal products of loops:

$$\Phi = c_0 + \sum_i c_i\ [\alpha_i] + \sum_{jk} c_{jk}\ [\alpha_j][\alpha_k]\ + \dots\ , \tag{2.8}$$

where the c's are arbitrary complex number and the α's are loops (see also [11]). The loop observable (2.5) has an immediate extension to this algebra as[5]

$$\mathcal{T}[\Phi] = c_0 + \sum_i c_i \mathcal{T}[\alpha_i] + \sum_{jk} c_{jk} \mathcal{T}[\alpha_j]\, \mathcal{T}[\alpha_k]\ + \dots \tag{2.9}$$

The algebra $\mathcal{A}^f[\mathcal{L}]$ contains the ideal

$$\mathcal{K} = \{\Phi \in \mathcal{A}^f[\mathcal{L}]\ \mid\ \mathcal{T}[\Phi] = 0\}, \tag{2.10}$$

and we define the carrier space $\mathcal{V}$ of the representation by

$$\mathcal{V} = \mathcal{A}^f[\mathcal{L}]/\mathcal{K}. \tag{2.11}$$

In other words, the state space of the loop representation is defined as the space of the equivalence classes of linear combinations of multiloops, under the equivalence defined by the Mandelstam relations

$$\Phi \sim \Psi \ \text{ if } \ \mathcal{T}[\Phi] = \mathcal{T}[\Psi], \tag{2.12}$$

namely by the equality of the corresponding holonomies [11].

There is natural basis in this linear space, denoted the spin network basis, which was introduced in [32], and developed in [33]. This is defined as follows. A spin network S is here a graph imbedded in the three dimensional space M, with a "color" (a positive integer) assigned to each link of the graph. Vertices with valence higher than three are (arbitrarily) expanded in tree-like "virtual" trivalent graphs and the "'virtual" edges are colored as well (see [16]). Colors satisfy a condition at the vertices: in a trivalent vertex, each color is not larger than the sum of the other two (Clebsh-Gordon condition), and the sum of the three colors is even.

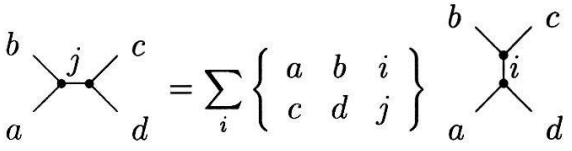
[5]The following formula corresponds to equation (2.15) in ref.[16]. However, equation (2.15) in [16] contains an additional (-2) factor in the first term. The (-2), and the motivation given in [16] for its introduction, are not correct. I thank Laszlo Szabados for pointing this out.

There exists a procedure to associate a linear combination of formal products of loops, and therefore an element of the quantum state space $\mathcal{V}$, to each such spin network. The procedure (introduced by Penrose [34]) consists in replacing each (real and virtual) edge colored p with p overlapping lines, joining these lines at the vertices and then anti-symmetrizing the lines in each (real and virtual) edge. One can then prove that the quantum states $|S\rangle$ obtained in this way form a basis in $\mathcal{V}$.

Finally, a scalar product is naturally defined over $\mathcal{V}$ (see [16], and below). We can complete in the Hilbert norm, obtaining the ("unconstrained" or "kinematical") Hilbert space of the quantum theory, which we denote as $\mathcal{H}$.

2.2 Structures in $\mathcal{H}$

The Hilbert space $\mathcal{H}$ has a rich structure that has been extensively explored. First of all, the spin network states satisfy the Kauffman axioms of the tangle theoretical version of recoupling theory [35] (in the "classical" case $A = -1$) at all the points (in 3d space) in which they meet [6]. For instance, weave consider a 4-valent intersection of four edges colored a, b, c, d. The color of the vertex is determined by expanding the 4-valent intersection into a trivalent tree; in this case, we have a single internal edge. The expansion can be done in different ways (by pairing edges differently). These are related to each other by the recoupling theorem of pg. 60 in Ref. [35]



(2.13)

where the quantities $\left\{ \begin{array}{ccc} a & b & i \\ c & d & j \end{array} \right\}$ are $su(2)$ six-j symbols (normalized as in [35]). Equation (2.13) follows just from the definitions given above. Recoupling theory provides a powerful computational tool in this context.

Since spin network states satisfy recoupling theory, they form a Temperley-Lieb algebra [35]. The scalar product in $\mathcal{H}$ is given by the Temperley-Lieb trace of the spin networks, or, equivalently by the Kauffman brackets, or, equivalently, by the chromatic evaluation of the spin network. Spin network states form an orthogonal base. See Ref. [16] for an extensive discussion of these relations.

Next, the space $\mathcal{H}$ can be constructed as the projective limit of a (projective) family of Hilbert spaces $\mathcal{H}_\Gamma$ of $SU(2)$ lattice gauge theories defined over arbitrary lattices Γ in three-space [36]. The space $\mathcal{H}_\Gamma$ naturally sits into the space $\mathcal{H}_{\Gamma'}$ when the graph Γ is a subgraph of Γ', and, correspondingly, the spaces $\mathcal{H}_\Gamma$ form a projective family.

[6] This fact is often misunderstood: recoupling theory lives in 2d and is associated by Kauffman to knot theory by means of the usual projection of knots from 3d to 2d. Here, the Kauffmann axioms are not satisfied at the intersections created by the 2d projection of the spin network, but only at the true intersections in 3d. See [16] for a detailed discussion.

Next, $\mathcal{H}$ can be viewed as the space of gauge-invariant functions over (the closure in a suitable norm of) the space $\mathcal{A}$ of the $SU(2)$ gauge connections, which are square integrable under the Ashtekar-Lewandowski-Baez measure $d\mu_{ALB}[A]$ [37]. $\mathcal{A}$ can be thought as a space of "distributional connections". The Ashtekar-Lewandowski-Baez measure is a diffeomorphism invariant measure over such space. (Or, equivalently, a "generalized measure" over the space of smooth connection [33].) The cylindrical functions over which the measure is constructed correspond precisely to the spin network states defined above.

The relation is as follows. When restricted to the (dense) subspace of $\mathcal{A}$ formed by smooth connections, the cylindrical function $\psi_S[A] = \langle A|S\rangle$ corresponding to a given spin network state $|S\rangle$ is formed by parallel propagators of the $SU(2)$ connection along the edges of S, in the representation $p/2$, where p is the color of the edge, contracted at the vertices by means of invariant tensors in the tensor product of the representations associated to the edges joining at the vertex. The colors of the vertex (namely the colors of the internal edges) label the independent invariant tensors[7]. This construction gives a rigorous meaning to the loop transform, which was used as an heuristic devise to build the loop representation in [8]. In fact, we can write, for every spin netwok s, and every state $\psi[A]$

$$\psi(S) = \langle S|\psi\rangle = \int d\mu_{ALB}[A]\ \bar{\psi}_S[A]\ \psi[A] \tag{2.14}$$

One can show that this equation defines a unitary mapping between the two presentations of $\mathcal{H}$: the "loop representation", in which one works in terms of the basis $|S\rangle$; and the "connection representation", in which one uses wave functionals $\psi[A]$.

For a recent discussion of the unitary equivalence between loop and connection representations see [39] and [40]. The relation between the two representations is also an implementation of the well known duality between $SU(2)$ representation theory and the combinatorics of planar loops. This duality has been much exploited in physical applications, and underlies all graphical methods for dealing with $SU(2)$ representation theory [41]. It was Penrose who first had the intuition that this mathematics could be relevant for describing the quantum properties of the geometry, and who gave the first version of spin network theory [34].

Finally, Ashtekar and Isham [38] have recovered the representation of the loop algebra by using C*-algebra representation theory: The space $\mathcal{A}/\mathcal{G}$, where $\mathcal{G}$ is the group of local $SU(2)$ transformations, is precisely the Guelfand spectrum of the abelian part of the loop algebra. One can show that this is a suitable norm closure of the space of smooth $SU(2)$ connections over physical space, modulo gauge transformations.

Thus, a number of powerful mathematical tools are at hand for dealing with nonperturbative quantum gravity. Some of these have already been extensively used in this context. These include: Penrose's spin network theory, $SU(2)$ representation theory, Kauffman tangle theoretical recoupling theory, Temperly-Liebb algebras, Gelfand's C^*algebra spectral representation theory, infinite dimensional measure theory and differential geometry over infinite dimensional spaces.

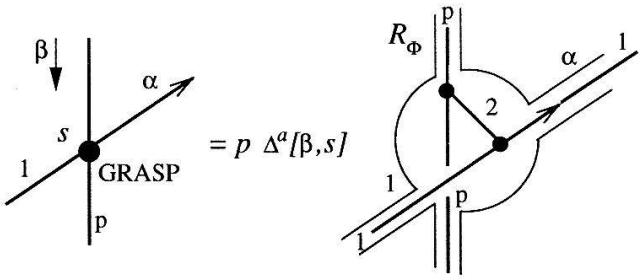
[7]Because a basis of invariant $SU(2)$ tensor on the tensor product of a finite number of irreps. is obtained by progressively decomposing tensor products or irreps. into irreps., two by two.

2.3 The representation

We now define the quantum operators, corresponding to the $\mathcal{T}$-variables, as linear operators on $\mathcal{H}$. These form a representation of the loop variables Poisson algebra. The operator $\hat{\mathcal{T}}[\alpha]$, acting on a state $\langle\Phi|$ simply adds a loop to $\langle\Phi|$:

$$\left\langle c_0 + \sum_i c_i\ [\alpha_i] + \sum_{ij} c_{ij}\ [\alpha_i][\alpha_j]\ + \ldots \right|\ \hat{\mathcal{T}}[\alpha] =$$
$$= \left\langle c_0[\alpha] + \sum_i c_i\ [\alpha_i][\alpha] + \sum_{ij} c_{ij}\ [\alpha_i][\alpha_j][\alpha] + \ldots \right| \quad . \tag{2.15}$$

(The consuetudinal bra notation is just a historical left-over from the period when the scalar product was not known.) Higher order loop operators are expressed in terms of the elementary "grasp" operation: acting on an edge with color p, the hand of the loop operator creates two "virtual" trivalent vertices, one on the spin-network state and one the loop of the operator. The two virtual vertices are joined by a virtual edge of color 2.



(2.16)

where we have introduced the elementary length l_0 by

$$l_0^2 = \hbar G = \frac{16\pi\hbar G_{\text{Newton}}}{c^3} = 16\pi\ l_{Planck}^2 \tag{2.17}$$

and

$$\Delta^a[\beta, s] = \int_\beta d\tau\ \dot{\beta}^a(\tau)\delta^3[\beta(\tau), s]. \tag{2.18}$$

The sign of the tangent of β in $\Delta^a[\beta, s]$ is determined by the orientation of β consistent with the positive-terms of the loop expansion of the spin network. Higher order loop operators act similarly. One can verify that these operators provide a representation of the classical Poisson loop algebra.

All the operators in the theory are then constructed in terms of these basics loop operators, in the same way in which in conventional QFT one constructs all operators, including the Hamiltonian, in terms of creation and annihilation operators. The construction of the composite operators requires the development of regularization techniques that can be used in the absence of a background metric. These have been introduced in [12] and developed in [4, 17, 16, 36]. I will illustrate these techniques below.

2.4 Diffeomorphism invariance

The next step in the construction of the theory is to factor away diffeomorphism invariance. This is a key step for two reasons. First of all, $\mathcal{H}$ is a "huge" non separable space. It is far "too large" for a quantum field theory. However, most of this redundancy is all gauge, and disappears when one solves the diffeomorphism constraint, defining the physical Hilbert space $\mathcal{H}_{Ph}$. This is the reason for which the loop representation, as defined here, is of great value in diffeomorphism invariant theories only.

The second reason is that $\mathcal{H}_{Ph}$ turns out to have a natural basis labeled by knots. More precisely by "s-knots". An s-knot s is an equivalence classes of spin networks S under diffeomorphisms. An s-knot is characterized by its "abstract" graph (defined only by the adjecency relations between edges and vertices), by the coloring, and by its knotting and linking properties, as in knot-theory.[8] Thus, the physical quantum states of the gravitational field turn out to be essentially classified by knot theory.

There are various equivalent way of obtaining $\mathcal{H}_{Ph}$ from $\mathcal{H}$. One can use regularization techniques for defining the quantum operator corresponding to the classical diffeomorphism constraint in terms of elementary loop operators, and then find the kernel of such operator. Equivalently, one can factor $\mathcal{H}$ by the natural action of the Diffeomorphism group that it carries. Namely

$$\mathcal{H}_{Ph} = \frac{\mathcal{H}}{Diff(M)}. \tag{2.19}$$

For a rigorous way for defining such a quotient of an Hilbert space by an infinite dimensional group, see [36] and references therein.

2.5 Dynamics

Finally, the definition of the theory is completed by giving the Hamiltonian constraint. A number of approaches to the definition of a Hamiltonian constraint have been attempted in the past, with various degrees of success. Recently, however, Thiemann has succeded in providing a regularization of the Hamiltonian constraint that yields a well defined, finite operator in $\mathcal{H}_{Ph}$. Thiemann's construction [30] is based on several clever ideas. I will not describe it here. Rather, I will sketch below the final form of the constraint (for the Lapse=1 case), following [44].

I begin with the Euclidean Hamiltonian constraint H_E. We have

$$\hat{H}|s\rangle = \sum_i \sum_{(IJ)} \sum_{\epsilon=\pm 1} \sum_{\epsilon'=\pm 1} A_{\epsilon\epsilon'}(p_i...p_n)\ \hat{D}_{i;(IJ),\epsilon\epsilon'}\ |s>\ . \tag{2.20}$$

Here i labels the vertices of the s-knot s; (IJ) labels couples of (distinct) edges emerging from i. $p_1...p_n$ are the colors the edges emerging from i. $\hat{D}_{i;(IJ)\epsilon\epsilon'}$ is the operator that acts

[8] Finite dimensional moduli spaces associated with high valence intersections appear [42]. Their physical relevance is unclear at this stage.

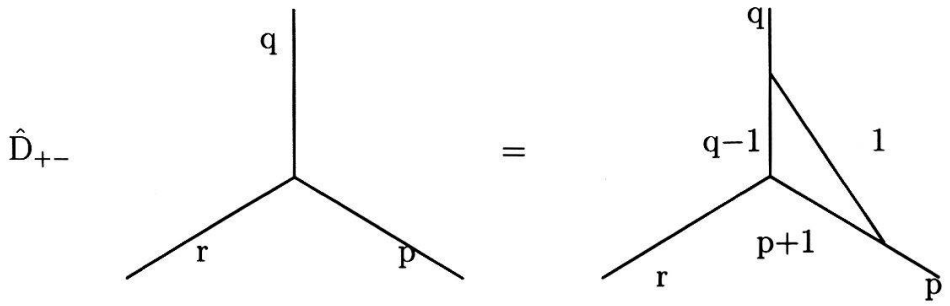


Figure 1: Action of $\hat{D}_{i;(IJ)\epsilon\epsilon'}$.

on an s-knot by: (i) creating two additional vertices, one along each of the two links I and J; (ii) creating a novel link, colored 1, joining these two nodes, (iii) assigning the coloring $p_I + \epsilon$ and, respectively, $q_J + \epsilon'$ to the links that join the new formed nodes with the node i. This is illustrated in Figure 1.

The coefficients $A_{\epsilon\epsilon'}(p_i...p_n)$, which are finite, can be expressed explicitly (but in a rather laborious way) in terms of products of linear combinations of $6-j$ symbols of $SU(2)$, following the techniques developed in detail in [16]. Some of these coefficients have been explicitly computed [43]. The Lorentzian Hamiltonian constraint is given by a similar expression, but quadratic in the $\hat{D}$ operators.

2.6 Developments

In the previous section, I have sketched the basic structure of the loop representation. This has been developed in a great number of directions. Without any ambition of completeness, I list below some of these developments.

- *Solutions of the Hamiltonian constraints.* One of the most surprising results of the theory is that it has been possible to find exact solutions of the Hamiltonian constraint. This follows from the key result that the action of the Hamiltonian constraints is non vanishing only over vertices of the s-knots [7, 8]. Therefore s-knots without vertices are physical states that solve the quantum Einstein dynamics. There is an infinite number of independent states of this sort, classified by conventional knot theory. The physical interpretation of these solutions is still rather obscure. But the issue has received much attention, and various other solutions have been found. See the recent review [15] and reference therein. See also [13, 14, 45, 46].

- *Time evolution. Strong field perturbation expansion. "Topological Feynman rules".* Trying to describe the temporal evolution of the quantum gravitational field by solving the Hamiltonian constraint yields the conceptually well-defined [47], but notoriously very non-transparent frozen-time formalism. An alternative is to study the evolution of the gravitational degrees of freedom with respect to some matter variable, coupled

to the theory, which plays the role of a phenomenological "clock". This approach has lead to the definition of a physical Hamiltonian [48], and to a preliminary investigation of the possibility of transition amplitudes between s-knot states, order by order in a (strong coupling) perturbative expansion [49]. In this context, diffeomorphism invariance, combined with the key result that the Hamiltonian constraint acts on vertices only, imply that the "Feynman rules" of such an expansion are purely topological and combinatorial.

- *Classical limit. Quantum states representing flat spacetime. Weaves. Discrete small scale structure of flat space.* The s-knots do not represent excitations of the quantum gravitational field over flat space, but rather over a "no-space", or $g_{\mu\nu} = 0$ solution. A natural problem is then how flat space (or any other smooth geometry) is represented in the theory. Notice that in a general relativistic context the Minkowski solution does not have all the properties of the conventional field theoretical vacuum. (In gravitational physics there is no real equivalent of the conventional vacuum, particularly in the spacially compact case.) One then expects that flat space is represented by some highly excited state in the theory. States in $\mathcal{H}$ that describe flat space when probed at low energy (large distance) have been studied in [4]. These have a discrete structure at the Planck scale. Furthermore, small excitations around such states have been considered in [50], where it is shown that $\mathcal{H}$ contains all "free graviton" physics, in a suitable approximation.

- *Fermions.* Fermions have been added to the theory [51]. Remarkably, all the important results of the pure GR case survive in the GR+fermions theory. Not surprisingly, fermions can be described as open ends of "open spin networks".

- *Maxwell.* The extension of the theory to the Maxwell field has been studied in [52].

- *Application to other theories.* The loop representation has been applied in various other contexts such as 2+1 gravity [53], some topological field theories, and others.

- *Lattice and simplicial models.* A number of very interesting discretized versions of the theory are being studied. See in particular [54].

- *Spectra of geometrical quantities. Area and volume.* Finally, the results that I consider most characteristic and potentially most fruitful regard spectral properties of geometrical quantities, such as area and volume of regions physically defined (say by matter). I will focus on these results in the next section.

3 Area

Consider a physical situation in which the gravitational field is interacting with some matter. We are interested in the area of a surface defined by the matter. For instance, imagine we are studying the explosion of a supernova. One second after the explosion, the matter of the supernova is approximately spherical, and defines a surface Σ: the surface of the star. The

physical area of Σ depends on the matter as well as on the metric, namely on the gravitational field. In a quantum theory of gravity, the gravitational field is a quantum field operator, and therefore we must describe the area of Σ in terms of a quantum observables described by an operator $\hat{A}$. We now ask what is the quantum operator $\hat{A}$ in nonperturbative quantum gravity.

Consider a 2d surface Σ imbedded in M with coordinates $\sigma^u = (\sigma^1, \sigma^2)$. We write $S : \Sigma \longrightarrow M, \sigma^u \longrightarrow x^a(\sigma)$. The metric and the normal one form on Σ are given by

$$g^\Sigma = S^\star\, g, \qquad g^\Sigma_{uv} = \frac{\partial x^a}{\partial \sigma^u}\frac{\partial x^b}{\partial \sigma^v} g_{ab}; \tag{3.1}$$

$$n_a = \frac{1}{2}\epsilon^{uv}\epsilon_{abc}\frac{\partial x^b}{\partial \sigma^u}\frac{\partial x^c}{\partial \sigma^v}. \tag{3.2}$$

The area of Σ is

$$\begin{aligned} A[\Sigma] &= \int_\Sigma d^2\sigma\, \sqrt{\det g^\Sigma} = \int_\Sigma d^2\sigma\, \sqrt{\frac{1}{2}\epsilon^{u\bar{u}}\epsilon^{\bar{v}\bar{v}} g^\Sigma_{uv} g^\Sigma_{\bar{u}\bar{v}}} \\ &= \int_\Sigma d^2\sigma\, \sqrt{n_a n_b \tilde{E}^{ai}\tilde{E}^b_i}, \end{aligned} \tag{3.3}$$

(On the role of played by surface area in the Ashtekar's formulation of GR, see [55].) We want to construct the quantum area operator $\hat{A}[\Sigma]$, namely a function of the loop representation operators whose classical limit is $A[\Sigma]$. Following conventional quantum field theoretical techniques, we deal with operator products by defining $\hat{A}[\Sigma]$ as a limit of regularized operators $\hat{A}_\epsilon[\Sigma]$ that do not contain operator products. The difficulty in the present context is to find a regularization that does not break general covariance. This can be achieved by a geometrical regularization [12].

Following [17], we begin by constructing a classical regularized expression for the area, namely a one parameter family of classical functions of the loop variables $A_\epsilon[\Sigma]$ which converges to the area as ϵ approaches zero.[9] Consider a small region Σ_ϵ of the surface Σ, whose coordinate area goes to zero with ϵ^2. For every s in Σ, the smoothness of the classical fields implies that $\tilde{E}^a(s) = \tilde{E}^a(x_I) + O(\epsilon)$, where x_I is an arbitrary fixed point in Σ_ϵ. Also, $U_\alpha(s,t)^B_A = \delta^B_A + O(\epsilon)$ for any $s, t \in \Sigma_I$ and α a (coordinate straight) segment joining s and t. It follows that to zeroth order in ϵ

$$\begin{aligned} \mathcal{T}^{ab}[\alpha_{st}](s,t) &= -\mathrm{Tr}\left[\tilde{E}^a(s) U_\alpha(s,t)\tilde{E}^b(t) U_\alpha(t,s)\right] \\ &= 2\tilde{E}^{ai}(x_I)\tilde{E}^b_i(x_I). \end{aligned} \tag{3.4}$$

Using this, we can write

$$\begin{aligned} \epsilon^4 \tilde{E}^{ai}(x_I)\tilde{E}^b_i(x_I) = \frac{1}{2}\int_{\Sigma_\epsilon} d^2\sigma\; n_a(\sigma) \int_{\Sigma_\epsilon} d^2\tau\; n_b(\tau) \\ \mathcal{T}^{ab}[\alpha_{\sigma\tau}](\sigma,\tau) + O(\epsilon), \end{aligned} \tag{3.5}$$

[9] I simplify here. For regularization that works in the general case, see [19]

where $\alpha_{\sigma\tau}$ is, say, a (coordinate) circular loop with the two points σ and τ on antipodal points. Next, consider the area of the full surface Σ. By definition of Riemann integral, (3.3) can be written as

$$\begin{aligned} A[\Sigma] &= \int_\Sigma d^2\sigma \sqrt{n_a n_b \tilde{E}^{ai}\tilde{E}^b_i} \\ &= = \lim_{\substack{N\to\infty \\ \epsilon\to 0}} \sum_{I_\epsilon} \epsilon^2 \sqrt{n_a(x_I) n_b(x_I) \tilde{E}^{ai}(x_I)\tilde{E}^b_i(x_I)} \end{aligned} \tag{3.6}$$

where, following Riemann we have partitioned the surface Σ in N small surfaces Σ_{I_ϵ} of coordinate area ϵ^2 and x_I is an arbitrary point in Σ_{I_ϵ}. Inserting (3.5) in (3.7), we obtain the desired regularized expression for the classical area, suitable to be promoted to a quantum loop operator

$$A[\Sigma] = \lim_{\epsilon\to 0} A_\epsilon[\Sigma] \ , \tag{3.7}$$

$$A_\epsilon[\Sigma] = \sum_{I_\epsilon} \sqrt{A^2_{I_\epsilon}} \ , \tag{3.8}$$

$$A^2_{I_\epsilon} = \frac{1}{2}\int_{\Sigma_{I_\epsilon}\otimes\Sigma_{I_\epsilon}} d^2\sigma d^2\tau \ n_a(\sigma) n_b(\tau) \ \mathcal{T}^{ab}[\alpha_{\sigma\tau}](\sigma,\tau). \tag{3.9}$$

Notice that the powers of the regulator ϵ in (3.5) and (3.7) combine nicely, so that ϵ appears in (3.7) only in the integration domains.

We are now ready to define the area operator:

$$\hat{A}[\Sigma] = \lim_{\epsilon\to 0} \hat{A}_\epsilon[\Sigma], \tag{3.10}$$

$$A_\epsilon[\Sigma] = \sum_{I_\epsilon} \sqrt{\hat{A}^2_{I_\epsilon}}, \tag{3.11}$$

$$\hat{A^2_{I_\epsilon}} = \frac{1}{2}\int_{\Sigma_{I_\epsilon}\otimes\Sigma_{I_\epsilon}} d^2\sigma d^2\tau \ n_a(\sigma) n_b(\tau) \ \hat{\mathcal{T}}^{ab}[\alpha_{\sigma\tau}](\sigma,\tau). \tag{3.12}$$

The meaning of the limit in (3.10) is discussed in detail in [16].

We now study the action of the area operator $\hat{A}[\Sigma]$ given in (3.10) on a spin network state $\langle S|$. We label by an index i the points where the spin network graph Γ_S and the surface Σ intersect. (Here we disregard spin networks that have a vertex lying on Σ or a continuous number of intersection points with Σ. The complete spectrum of the area, including these cases is given in [18, 19].)

For small enough ϵ, each intersection i will lie inside a distinct Σ_{I_ϵ} surface. Let us call Σ_{i_ϵ} the surface containing the intersection i (at every fixed ϵ), and e_i the edge through the intersection i. Notice that $\langle S|\hat{A}^2_{\Sigma_{I_\epsilon}}$ vanishes for all surfaces I_ϵ except the ones containing intersections. Thus the sum over surfaces $\sum_{I_\epsilon}$ reduces to a sum over intersections. Bringing the limit inside the sum and the square root, we can write

$$\langle S|\hat{A}[\Sigma] = \sum_{i\in\{S\cap\Sigma\}} \langle S|\sqrt{\hat{A}^2_i} \tag{3.13}$$

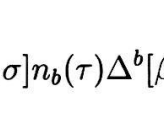
$$\hat{A}^2_i = \lim_{\epsilon\to 0} \hat{A}^2_{i_\epsilon} \tag{3.14}$$

For finite ϵ, the state $\langle S|\hat{A}^2_{i_\epsilon}$ has support on the union of the graphs of S and the graph of the loop $\alpha_{\sigma\tau}$ in the argument of the operator (3.12). But the last converges to a point on Γ_S as ϵ goes to zero. Therefore

$$\lim_{\epsilon\to 0}\Gamma_{\langle S|\hat{A}^2_{i_\epsilon}} = \Gamma_S. \tag{3.15}$$

The operator $\hat{A}[\Sigma]$ does not affect the graph of $\langle S|$. Next, we have to compute the combinatorial part of the action of the operator. By equation (3.13), this is given by a sum of terms, one for each $i \in \{S \cap \Sigma\}$. Consider one of these terms. By definition of the $\hat{T}$ loop operators and of the grasp operation (Section 3), this is obtained by inserting two trivalent intersections on the spin network edge e_i, connected by a new edge of color 2. (The circle $\Gamma_{\alpha_{\sigma\tau}}$ has converged to a point on e_i; in turn, this point is then graphically expanded following back and forward a segment connecting the two intersections. By indicating the representation of the spin network simply by means of its e_i edge, we thus have

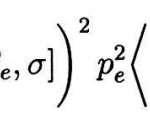
$$\begin{aligned}\langle|^{p_e} \mid \hat{A}^2_{i_\epsilon} &= \frac{1}{2}\int_{\Sigma_{i_\epsilon}\otimes\Sigma_{i_\epsilon}} d^2\sigma d^2\tau\; n_a(\sigma)n_b(\tau)\;\langle|^{p_e} \mid \hat{T}^{ab}[\alpha_{\sigma\tau}](\sigma,\tau) \\ &= -\frac{l_0^4}{2}\int_{\Sigma_{i_\epsilon}\otimes\Sigma_{i_\epsilon}} d^2\sigma d^2\tau\; n_a(\sigma)\Delta^a[\beta_e,\sigma]n_b(\tau)\Delta^b[\beta_e,\tau]\; p_e^2 \Big\langle \end{aligned} \tag{3.16}$$



$$\Big|$$

where we have already taken the limit (inside the integral) in the state enclosed in the brackets $\langle\ |$. Notice that this does not depend on the integration variables anymore, because the loop it contains does not represent the grasped loop for a finite ϵ. Notice also that the two integrals are independent, and equal. Thus, we can write

$$\langle|^{p_e} \mid \hat{A}^2_{i_\epsilon} = -\frac{l_0^4}{2}\left(\int_{\Sigma_{I_\epsilon}} d^2\sigma\; n_a(\sigma)\Delta^a[\beta_e,\sigma]\right)^2 p_e^2 \Big\langle \tag{3.17}$$



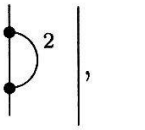
$$\Big|$$

The parenthesis is easy to compute. Using (2.18), it becomes the analytic form of the intersection number between the edge and the surface

$$\begin{aligned}\int_{\Sigma_{i_\epsilon}} d^2\sigma\; n_a(\sigma)\Delta^a[\beta_e,\sigma] &= \int_{\Sigma_{i_\epsilon}} d^2\sigma\; n_a(\sigma)\int_{\beta_e} d\tau\; \dot{\beta}^a_e(\tau)\delta^3[\beta_e(\tau),s] \\ &= \pm 1,\end{aligned} \tag{3.18}$$

where the sign, which depends on the relative orientation of the loop and the surface, becomes then irrelevant because of the square. Thus

$$\langle|^{p_e} \mid \hat{A}^2_i = -\frac{l_0^4}{2}\, p_e^2 \Big\langle \tag{3.19}$$



$$\Big|,$$

where we have trivially taken the limit (3.14), since there is no residual dependence on ϵ. We have now to express the tangle inside the bracket in terms of (an edge of) a spin network state. But tangles satisfy recoupling theory, and we can therefore use the formula (E.8) in

the appendix of [16], obtaining

$$\langle|^{p_e} \mid \hat{A}^2_{i_\epsilon} = -l_0^4\, p_e^2\, \frac{\theta(p_e, p_e, 2)}{2\Delta_{p_e}}\, \langle|^{p_e} \mid =$$
$$= l_0^4\, \frac{p_e(p_e+2)}{4}\, \langle|^{p_e} \mid = l_0^4\, \frac{p_e}{2}\left(\frac{p_e}{2}+1\right)\, \langle|^{p_e} \mid.$$

The square root in (3.13) is now easy to take because the operator $\hat{A}^2_i$ is diagonal.

$$\langle|^{p_e} \mid \hat{A}_i = \langle|^{p_e} \mid \sqrt{\hat{A}^2_i} = = \sqrt{l_0^4\, \frac{p_e}{2}\left(\frac{p_e}{2}+1\right)}\, \langle|^{p_e} \mid. \tag{3.20}$$

Inserting in the sum (3.13), we obtain the final result

$$\langle S|\ \hat{A}[\Sigma] = \left(\frac{l_0^2}{2} \sum_{i \in \{S \cap \Sigma\}} \sqrt{p_i(p_i+2)}\right)\ \langle S| \tag{3.21}$$

This result shows that the spin network states (with a finite number of intersection points with the surface and no vertices on the surface) are eigenstates of the area operator. The corresponding spectrum is labeled by multiplets $\vec{p} = (p_1, ..., p_n)$ of positive half integers, with arbitrary n, and given by

$$A_{\vec{p}}[\Sigma] = \frac{l_0^2}{2} \sum_i \sqrt{p_i(p_i+2)}. \tag{3.22}$$

Shifting from color to spin notation, we have

$$A_{\vec{j}}[\Sigma] = l_0^2 \sum_i \sqrt{j_i(j_i+1)}, \tag{3.23}$$

where $j_1, ..., j_n$ are half integer. This expression reveals the $SU(2)$ origin of the spectrum.

A similar result has been obtained for the volume.

4 Two applications

The first hint on the thermodynamical behavior of black holes comes from classical general relativity. Hawking's theorem [56] tells us that the area of the event horizon of a black hole cannot decrease in time, in classical general relativity. In ref. [26], Bekenstein speculated that one can associate an entropy $S(A)$ to a Schwarzschild black hole of surface area A, where

$$S = c\, \frac{k}{\hbar G}\, A \tag{4.1}$$

(c is a constant of the order of unity, k the Boltzman constant, and I put the speed of light equal to one). Bekenstein provided a number of physical arguments supporting this idea;

but the reaction of the physicists community was cold, mainly due to the fact that since the black hole area A is connected to the black hole energy M by

$$M = \sqrt{\frac{A}{16\pi G^2}}, \tag{4.2}$$

the standard thermodynamical relation $T^{-1} = k\, dS/dE$ would imply the existence of a black hole temperature

$$T = \frac{\hbar}{c32\pi kGM}, \tag{4.3}$$

and therefore in vacuum the black hole should emit thermal radiation at this temperature: a result difficult to believe. However, shortly after Bekenstein's suggestion, Hawking [27] derived black hole emission just from quantum field theory in curves spacetime. Hawking computed the emission temperature to be

$$T = \frac{\hbar}{8\pi kGM}, \tag{4.4}$$

which beautifully supports Bekenstein's speculation, and fixes the constant c at

$$c_{Hawking} = \frac{1}{4}. \tag{4.5}$$

Hawking's result opens many problems. I will consider two of these problems. First, in Hawking's derivation the quantum properties of gravity are neglected. Are these affecting the result? Second, in general we understand macroscopical entropy in statistical mechanical terms as an effect of microscopical degrees of freedom. What are the microscopical degrees of freedom responsible for (4.1)? Can one derive (4.1) from first principles? Clearly a complete answer of these questions requires a quantum theory of gravity.

4.1 The Bekenstein-Mukhanov effect

Recently, Bekenstein and Mukhanov [23] have suggested that the thermal nature of Hawking's radiation may be affected by quantum properties of gravity (For a review of earlier suggestions in this direction, see [57]). Bekenstein and Mukhanov observe that in most approaches to quantum gravity the area can take only quantized values [58]. Since the area of the black hole surface is connected to the black hole mass, black hole mass is likely to be quantized as well. The mass of the black hole decreases when radiation is emitted. Therefore emission happens when the black hole makes a quantum leap from one quantized value of the mass (energy) to a lower quantized value, very much as atoms do. A consequence of this picture is that radiation is emitted at quantized frequencies, corresponding to the differences between energy levels. Thus, quantum gravity implies a discretized emission spectrum for the black hole radiation.

By itself, this result is not physically in contradiction with Hawking's prediction of a continuous thermal spectrum. To understand this, consider the black body radiation of

a gas in a cavity, at high temperature. This radiation has a thermal Planckian emission spectrum, essentially continuous. However, radiation is emitted by elementary quantum emission processes yielding a discrete spectrum. The solution of the apparent contradiction is that the spectral lines are so dense in the range of frequencies of interest, that they give rise –effectively– to a continuous spectrum. Does the same happen for a black hole?

In order to answer this question, we need to know the energy spectrum of the black hole, which is to say, the spectrum of the Area. Bekenstein and Mukhanov pick up a simple ansatz: they assume that the Area is quantized in multiple integers of an elementary area A_0. Namely, that the area can take the values

$$A_n = nA_0, \tag{4.6}$$

where n is a positive integer, and A_0 is an elementary area of the order of the Planck Area

$$A_0 = \alpha\hbar G, \tag{4.7}$$

where α is a number of the order of unity (G is Newton's constant and $c = 1$). Ansatz (4.6) is reasonable; it agrees, for instance, with the partial results on eigenvalues of the area in the loop representation given in [4], and with the idea of a quantum picture of a geometry made by elementary "quanta of area". Since the black hole mass is related to the area by (4.2), it follows from this relation and the ansatz (4.6) that the energy spectrum of the black hole is given by

$$M_n = \sqrt{\frac{n\alpha\hbar}{16\pi G}}. \tag{4.8}$$

Consider an emission process in which the emitted energy is much smaller than the mass M of the black hole. From (4.8), the spacing between the energy levels is

$$\Delta M = \frac{\alpha\hbar}{32\pi GM}. \tag{4.9}$$

From the quantum mechanical relation $E = \hbar\omega$ we conclude that energy is emitted in frequencies that are integer multiple of the fundamental emission frequency

$$\bar{\omega} = \frac{\alpha}{32\pi GM}. \tag{4.10}$$

This is the fundamental emission frequency of Bekenstein and Mukhanov [23] (they assume $\alpha = 4\ln 2$). Bekenstein and Mukhanov proceed in [23] by showing that the emission amplitude remains the same as the one in Hawking's thermal spectrum, so that the full emission spectrum is given by spectral lines at frequencies multiple of $\bar{\omega}$, whose envelope is Hawking's thermal spectrum.

As emphasized by Smolin in [57], however, the Bekenstein-Mukhanov spectrum is drastically different than the Hawking spectrum. Indeed, the maximum of the Planckian emission spectrum of Hawking's thermal radiation is around

$$\omega_H \sim \frac{2.82kT_H}{\hbar} = \frac{2.82}{8\pi GM} = \frac{2.82\cdot 4}{\alpha}\bar{\omega} \approx \bar{\omega}. \tag{4.11}$$

That is: the fundamental emission frequency $\bar{\omega}$ is of the same order as the maximum of the Planck distribution of the emitted radiation. It follows that there are only a few spectral lines in the regions where emission is appreciable. Therefore the Bekenstein-Mukhanov spectrum is drastically different than the Hawking spectrum: the two have the same envelope, but while Hawking spectrum is continuous, the Bekenstein-Mukhanov spectrum is formed by just a few lines in the interval of frequencies where emission is appreciable. Notice that such a discretization of the emission spectrum is derived by Bekenstein and Mukhanov on purely kinematical grounds, that is using only the (assumed) spectral properties of the area. To emphasize this fact, we will denote it as the kinematical Bekenstein-Mukhanov effect.

This result is of great interest because, in spite of its weakness, black hole radiation is still much closer to the possibility of (indirect) investigation than any quantum gravitational effect of which we can think. Thus, a clear quantum gravitational signature on the Hawking spectrum is a very interesting effect. Is this Bekenstein-Mukhanov effect credible?

As first suggested in [57], and, independently, by Brügmann, one may use loop quantum gravity to check the Bekenstein-Mukhanov result, by replacing the naive ansatz (4.6) with the precise spectrum computed in loop quantum gravity.

Consider a surface Σ –in the present case, the event horizon of the black hole–. The area of Σ can take only a set of quantized values. These quantized values are labeled by unordered n-tuples of positive integers $\vec{p} = (p_1, ..., p_n)$ of arbitrary length n. The spectrum is given in (3.22). If we disregard for a moment the term $+1$ under the square root in (3.22), we obtain immediately the ansatz (4.6), and thus the Bekenstein-Mukhanov result. However, the $+1$ is there. Let us study the consequences of its presence. First, let us estimate the number of Area eigenvalues between the value $A >>> l_0$ and the value $A + dA$ of the Area, where we take dA much smaller than A but still much larger than l_0. Since the $+1$ in (3.22) affects in a considerable way only the terms with low p_i, we can neglect it for a rough estimate. Thus, we must estimate the number of unordered strings of integers $\vec{p} = (p_1, ..., p_n)$ such that

$$\sum_{i=1,n} p_i = \frac{A}{8\pi\hbar G} >> 1. \tag{4.12}$$

This is a well known problem in number theory. It is called the partition problem. It is the problem of computing the number N of ways in which an integer I can be written as a sum of other integers. The solution for large I is a classic result by Hardy and Ramanujan [59]. According to the Hardy-Ramanujan formula, N grows as the exponent of the square root of I. More precisely, we have for large I that

$$N(I) \sim \frac{1}{4\sqrt{3}I} e^{\pi\sqrt{\frac{2}{3}I}}. \tag{4.13}$$

Applying this result in our case we have that the number of eigenvalues between A and $A+dA$ is

$$\rho(A) \approx e^{\sqrt{\frac{\pi A}{12\hbar G}}}. \tag{4.14}$$

Now, because of the presence of the $+1$ term, eigenvalues will overlap only accidentally: generically all eigenvalues will be distinct. Therefore, the average spacing between eigenvalues

decreases exponentially with the inverse of the square of the area. This result is to be contrasted with the fact that this spacing is constant and of the order of the Planck area in the case of the naive ansatz (4.6). This conclusion empties the Bekenstein-Mukhanov argument. Indeed, the density of the energy levels becomes

$$\rho(M) \approx e^{\sqrt{\frac{4\pi G}{3\hbar}}M}, \tag{4.15}$$

and therefore the spacing of the energy levels decreases *exponentially* with M. It follows that for a macroscopical black hole the spacing between energy levels is infinitesimal, and thus the spectral lines are virtually dense in frequency. We effectively recover in this way Hawking's thermal spectrum (except, of course, in the case of a Planck scale black hole). A weaker but rigorous lower bound on the density of eigenvalues, consistent with the argumented given here, is given in [18]. The conclusion is that the Bekenstein-Mukhanov effect disappears if we replace the naive ansatz (4.6) with the spectrum (3.22) computed from loop quantum gravity. More generally, the kinematical Bekenstein-Mukhanov effect is strongly dependent on the peculiar form of the naive ansatz (4.6), and it is not robust. In a sense, this is a pity, because we loose a possible window on quantum geometry.

Mukhanov and, independently, Smolin have noticed that the possibility is still open for the existence of a "dynamical" Bekenstein-Mukhanov effect [60]. For instance, transitions in which a single Planck unit of area is lost could be strongly favored by the dynamics. To explore if this is the case, one should make use of the full machinery of quantum gravity, for instance by computing transition probabilities between horizon's area eigenstates induced in a first order perturbation expansion by the coupling between the area of the horizon and a surrounding radiation field. This could perhaps be done following the lines of Ref. [44].

The conclusion is that the argument for the discretization of the black hole emission spectrum given by Bekenstein and Mukhanov is not valid, if we use quantitative result from loop quantum gravity. As emphasized by Mukhanov, this fact does not prove that the spectrum is indeed continuous, since a discretization could be still be consequence of other (dynamical) reasons.

4.2 Black Hole Entropy from Loop Quantum Gravity

Finally, I present a derivation [24, 25] of the Bekenstein-Hawking expression (4.1) for the entropy of a Schwarzschild black hole of surface area A via a statistical mechanical computation [61]. The strategy I follow is based on the idea that the entropy of the hole originates from the microstates of the horizon that correspond to a given macroscopic configuration.

This idea was first suggested in a seminal work by York [62]. York notices that the hole's radiance implies that the (macroscopic) event horizon is located slightly inside the quasistatic timelike limit-surface, leaving a thin shell between the two, which he proposes to interpret as the region over which the microscopic horizon fluctuates. He interprets these fluctuations as zero point quantum fluctuations of the horizon's quasinormal modes, and, by identifying the thermal energy of these oscillations with the shell's ("irreducible") mass, he is able to recover

Hawking's temperature. I take two essential ideas from York's work: that the source of the hole entropy is in the degrees of freedom associated with the fluctuations of the shape of the (microscopic) horizon; and that the quasilocal measure of mass-energy governing energetic exchanges between the horizon and its surroundings can be taken as the Christodoulou-Ruffini [63] "irreducible mass"

$$M_{CR} = \sqrt{\frac{A}{16\pi G^2}}. \tag{4.16}$$

Can we replace York's perturbative semiclassical approach with a direct calculation within nonperturbative quantum gravity?

The relevance of horizon's surface degrees of freedom for the entropy has been recently explored from various perspectives [64]. (See also [65] for an attempt to use the "membrane paradigm" [66]: interactions of a black hole with its surroundings can be described in terms of a fictitious physical membrane located close to the horizon). An approach strictly related to the one I am going to describe has been suggested in Refs. [67], where it is argued that a physical split of a gauge system gives rise to boundary degrees of freedom, since the boundary breaks the gauge group. Using this idea the Bekenstein-Hawking formula can be derived, by counting boundary states, in 3-d gravity. The relation is the following. In GR, the broken component of the gauge group includes diffeomorphisms that move the surface, and the boundary degrees of freedom can probably be viewed as fluctuations of the horizon.

Consider a physical system containing a non-rotating and non-charged black hole (say a collapsed star) as well as other physical components such as dust, gas or radiation, which we denote collectively (improperly) as "matter". We are interested in the statistical thermodynamics of such a system. A key observation is that because of Einstein's equations the microscopic time-dependent inhomogeneities of the matter distribution generate time-dependent "microscopic" inhomogeneities in the gravitational field as well. One usually safely disregards these ripples of the geometry. For instance, we say that the geometry over the Earth's surface is Minkowski (or Schwarzschild, due to the Earth gravitational field), disregarding the inhomogeneous time-dependent gravitational field generated by each individual fast moving air molecule. The Minkowski geometry is therefore a "macroscopic" coarse-grained average of the microscopic gravitational field surrounding us. However, in a statistical-thermodynamical treatment, these fluctuations should not be disregarded, because they are precisely the sources of the thermal behavior.

Statistical thermodynamics is based on the distinction between the macroscopic state of a system, determined by coarse-grained averaged physical quantities, and its macroscopic state determined by a (hypothetical) complete description of the system's dynamics. A system in equilibrium at a finite temperature T is macroscopically stationary. However, its microstate fluctuates over microscopic non-stationary configurations. The family of the microstates over which the system fluctuates when in a given macrostate form the statistical "ensemble" associated to the given macrostate. For instance, the macrostate of a gas in thermal equilibrium in a box is time-independent and spatially homogeneous, while the microstates in the corresponding ensemble are individually time dependent and non-homogeneous. Thus, we must have *two descriptions* of a physical black hole interacting with surrounding matter at finite

temperature. The macroscopic description is a stationary coarse grained description in which inhomogeneities are smoothed out. The microscopic description does not neglect the minute thermal motions.

Macroscopically, a non-charged and non-rotating hole is described by a stationary metric with non-charged and non-rotating event horizon. There is only a one-parameter family of solutions of Einstein equations with such properties: Schwarzschild with mass M, and corresponding event-horizon area $A = 16\pi G^2 M^2$. Therefore in a thermal context the Schwarzschild metric represents the coarse grained description of a microscopically fluctuating geometry. Microscopically the gravitational field is non-stationary (because it interacts with non-stationary matter) and non-spherically symmetric (because matter distribution is spherically symmetric on average only, and not on individual microstates). Its microstate, therefore is *not* given by the Schwarzschild metric, but by some complicated time-dependent non-symmetric metric.

I am convinced that taking such time-dependent non-symmetric microstates of the geometry into account is essential for a statistical understanding of the thermal behavior of black holes – as it is in understanding the thermal properties of any other system. Searching for a derivation of black hole thermodynamics from properties of stationary or symmetric metrics alone is like trying to derive the thermodynamics of an ideal gas in a spherical box just from spherically symmetric motions of the molecules.

Thus, consider the microstate of our system. Let us foliate spacetime with a family of spacelike surfaces Σ_t, labeled by a time coordinate t. The intersection h_t between the surface Σ_t and the future boundary of the past of future null-infinity defines the instantaneous (microscopic) configuration of the event horizon at time t. Thus, h_t is a closed 2-d surface immersed in Σ_t. For most times, this microscopic configuration of the event horizon is not spherically symmetric. Let us denote by g_t the intrinsic and extrinsic geometry of the horizon h_t. Let $\mathcal{M}$ be the space of all possible (intrinsic and extrinsic) geometries of a 2-d surface. As t changes, the (microscopic) geometry of the horizon changes. Thus, g_t wanders in $\mathcal{M}$ as t changes.

I now recall some standard techniques in statistical mechanics in a form that can be applied to our system. Consider a thermodynamical system $\mathcal{S}$, say an ideal gas in a isolated box. Consider an equilibrium macrostate of $\mathcal{S}$. Under suitable ergodicity conditions, the microstate of the system changes freely subjected to global conservation laws only. If the system is conservative and energy is the only conserved quantity, then the system will wander in the entire region of its phase space defined by a given total energy. Next, we can ideally split $\mathcal{S}$ into two subsystems $\mathcal{S}_1$ and $\mathcal{S}_2$, say two regions of the box, separated by a thin film. We are interested in studying the thermal interactions between the two subsystems. One approach is provided by the microcanonical point of view. Let us *ideally* isolate the subsystem $\mathcal{S}_1$. Namely let us momentarily assume that it cannot exchange heat. Let E_1 be its energy, and $S_1(E_1)$ its entropy, defined as the number of microstates that have energy E_1. We now relax the assumption that heat cannot be exchanged, and consider the full system $\mathcal{S}$. If a small amount of heat dQ is transferred from $\mathcal{S}_1$ to $\mathcal{S}_2$ the number of states available to $\mathcal{S}_1$ decreases by an amount $(dS_1/dE_1)dQ$ and the number of microstates available to $\mathcal{S}_2$ increases

by an amount $(dS_2/dE_2)dQ$. The total number of microstates available to $\mathcal{S}$ changes by

$$\delta N = \left(\frac{dS_2}{dE_2} - \frac{dS_1}{dE_1}\right) dQ. \tag{4.17}$$

From the assumption that the equilibrium macroscopical configuration is the one to which most microstates correspond, it follows that at equilibrium no small heat transfer dQ may increase the total number of available microstates, and therefore

$$\frac{dS_2}{dE_2} = \frac{dS_1}{dE_1}. \tag{4.18}$$

Namely, the temperatures of the two systems are equal.

Let us apply these ideas to our system. Consider our system as formed by two subsystems: the black hole and the rest. We want to associate an entropy S to the black hole, where S counts the number of microstates over which the hole may fluctuate in an ideal situation in which no heat (energy) is exchanged between the hole and its surroundings. The precise specification of this ensemble of microstates is crucial, and I now discuss it in detail.

First of all, as already noticed microscopic configurations do not need to be individually spherically symmetric. Second, only configurations of the hole itself, and not the configurations of the surrounding geometry, should affect the hole's entropy. Thus, we must focus on the state of the hole alone. Next, we are considering the thermodynamic behavior of a system containing the hole. This behavior cannot be affected by the hole's interior. The black hole interior may be in one of an infinite number of states indistinguishable from the outside. For instance, the black hole interior may (in principle) be given by a Kruskal spacetime; so that on the other side of the hole there is another "universe" (say spatially compact, if not for the hole) possibly with billions of galaxies. This potentially infinite number of such internal states does not affect the interaction of the hole with its surroundings and is irrelevant here, because it cannot affect the energetic exchanges between the hole and the outside, which are the ones that determine the entropy.

Therefore we are only interested in configurations of the hole *that have (microscopically) distinct effects on exterior of the hole.* From the exterior, the hole is completely determined by the geometrical properties of its surface. Thus, the entropy relevant for the thermodynamical description of the thermal interaction of the hole with its surroundings is entirely determined by the state of the gravitational field (of the geometry) on the black hole surface, namely by g_t.

Next, we have to determine the "ensemble" of the microstates g_t over which the hole may fluctuates under the ideal hypothesis of no heat exchange. In conventional statistical thermodynamics, one assumes that the only conserved quantity is energy, and the microcanonical ensemble is determined by fixing energy. Here, however, there is no obvious candidate for a notion of a conserved energy that could be used.

A physical observation that leads us to the solution of this problem is that if energy flows into the black hole then its area increases, while if the black hole radiates away energy (via

Hawking's radiation), then its area decreases. Therefore we are lead to the idea that the (ideal) situation of no heat (energy) exchange is the evolution at fixed horizon's area. Thus, following York, we take the Christodoulou-Ruffini quasi-local "irreducible mass" (4.16) as the relevant energy in this context (here A is the area of h_t); and we define the ensemble as the set of g_t in $\mathcal{M}$ with the same M_{CR}, namely with the same area A.

There is a number of reasons supporting the choice of this ensemble. First, M_{CR} is geometrically well defined, governs the hole's energy exchanges, and agrees with the macroscopic black hole energy. Second, the ensemble must contain reversible paths only. In the classical theory these conserve area (Hawking theorem [56]). Quantum theory allows classically forbidden energy exchanges with the exterior (Hawking radiance), but it is unlikely, we believe, that it would allow a nonreversible evolution of the horizon to become reversible without energy exchange with the exterior. Third, we may reason backward and let the thermodynamics indicate us the correct ensemble (which is how classical ensembles were first found). In this context, it perhaps worthwhile recalling that difficulties to rigorously justifying a priori the choice of the ensemble plague conventional thermodynamics anyway.

Summarizing, we are interested in counting the number $N(A)$ of states of the geometry g_t of a surface h_t of area A, where different regions of h_t are distinguishable from each other. The above discussion indicates then that $S(A) = k \ln N(A)$ is the entropy we should associate to the horizon in order to describe its thermal interactions with its surroundings. This "number" $N(A)$ meaningless in the classical theory. It is a this point only that we resort to the quantum theory. As the entropy of the electromagnetic field in a cavity is well defined only if we take quantum theory into account, similarly we may expect that the number of states $N(A)$ will be well defined in a quantum theory of gravity. The problem is thus to count the number of (orthogonal) quantum states of the geometry of a two dimensional surface, having total area A. The problem is now well defined, and can be translated into a direct computation.

If a surface Σ is given, its geometry is determined by its intersections with the s-knot. Intersections are of three types: (a) an edge crosses the surface; (b) a vertex lies on the surface; (c) a finite part of the s-knot lies on the surface. Intuitively, type (a) is the only "generic" case, and we should disregard states of type (b) and (c). Ashtekar has suggested a argument for neglecting type (b) and (c) intersections [68]: we wish to describe the geometry of a fluctuating surface Σ as observed from the exterior, and we expect the state of its geometry to be stable under infinitesimal deformations of Σ. We may thus consider the surface as the limit of a sequence of surfaces Σ_ϵ, and its state as the (Hilbert norm) limit of the states of Σ_ϵ. Clearly, states of type (b) and (c) cannot appear in this way, and therefore we have to restrict our computation to states having intersections of type (a) only [69]. The quantum geometry on the surface is then determined by the *ordered* n-tuples of integers

$$\vec{p} = (p_1, ..., p_n) \tag{4.19}$$

that form the colors of the edges of type (a) intersections.

Notice that in the previous section we were interested in counting the density of the eigenvalues of the area (because these determine the density of the lines in the emission spectrum). While here we are interested in counting the density of the eigenstates. Thus,

we must take the degeneracy of each eigenspace into account. n-tuples that differ from each other in the ordering yield of course the same total area. Therefore they should be considered indistinguishable in counting eigenvalues. On the other side they label distinct states.

One may be tempted to observe that such states can be transformed into each other by diffeomorphisms, and therefore should not be considered distinguishable. However, this observation is not correct. The point is that physical states are defined as equivalence classes under diffeomorphism of the full space, not the surface alone. To understand this point, let us consider a simplified analogy: Consider a set A, a set B, and a group G that acts (freely) on A and on B. Then G acts on $A \times B$. What is the space $\frac{A\times B}{G}$? One may be tempted to say that it is (isomorphic to) $\frac{A}{G} \times \frac{B}{G}$, but a moment of reflection shows that this is not correct and the correct answer is

$$\frac{A \times B}{G} \sim \frac{A}{G} \times B. \tag{4.20}$$

If G does not act freely over A, we have to divide B by the stability groups of the elements of A. Now, imagine that A is the space of the states of the exterior of the black hole, B the space of the states of the black hole, and G the diffeomorphism group of the horizon. Then we see that we must not divide B by the diffeomorphisms of the surface, but only by those diffeomorphisms that leave the rest of the spin network invariant. As far as the state on the surface is concerned, this amounts to restrict to diffeomorphisms that do not mix the intersections between the spin network and the surface. Therefore n-tuples with different ordering must be considered as distinct. Physically, this correspond to the fact that different locations in which the spin network punctures the surface can be distinguished from each other in terms of the external state of the gravitational field. For a more precise version of these remarks, see [70].

Thus, our task is reduced to the task of counting the ordered n-tuples of integers $\vec{p}$ such that (3.3). More precisely, we are interested in the number of microstates (n-tuples $\vec{p}$) such that the l.h.s of (3.3) is between A and $A + dA$, where $A >> \hbar G$ and dA is much smaller than A, but still macroscopic.

Let $M = A/8\pi\hbar G$, and let $N(M)$ be the number of ordered n-tuples $\vec{p}$, with arbitrary n, such that

$$\sum_{i=1,n} \sqrt{p_i(p_i+2)} = M. \tag{4.21}$$

First, we over-estimate $M(N)$ by approximating the l.h.s. of (4.21) dropping the +2 term under the square root. Thus, we want to compute the number $N_+(M)$ of ordered n-tuples such that

$$\sum_{i=1,n} p_i = M. \tag{4.22}$$

The problem is an exercise in combinatorics. It can be solved, for instance, by noticing that if $(p_1, ..., p_n)$ is a partition of M (that is, it solves (4.22)), then $(p_1, ..., p_n, 1)$ and $(p_1, ..., p_n + 1)$ are partitions of $M + 1$. Since all partitions of $M + 1$ can be obtained in this manner, we have

$$N_+(M+1) = 2N_+(M). \tag{4.23}$$

Therefore

$$N_+(M) = C\ 2^M. \tag{4.24}$$

Where C is a constant. In the limit of large M we have

$$\ln N_+(M) = (\ln 2)\ M. \tag{4.25}$$

Next, we under-estimate $M(N)$ by approximating (4.21) as

$$\sqrt{p_i(p_i+2)} = \sqrt{(p_i+1)^2 - 1} \approx (p_i + 1). \tag{4.26}$$

Thus, we wish to compute the number $N_-(M)$ of ordered n-tuples such that

$$\sum_{i=1,n} (p_i + 1) = M. \tag{4.27}$$

Namely, we have to count the partitions of M in parts with 2 or more elements. This problem can be solved by noticing that if $(p_1, ..., p_n)$ is one such partition of M and $(q_1, ..., q_m)$ is one such partition of $M-1$, then $(p_1, ..., p_n + 1)$ and $(q_1, ..., q_m, 2)$ are partitions of $M+1$. All partitions of $M+1$ in parts with 2 or more elements can be obtained in this manner, therefore

$$N_-(M+1) = N_-(M) + N_-(M-1). \tag{4.28}$$

It follows that

$$N_-(M) = Da_+^M + Ea_-^M \tag{4.29}$$

where D and E are constants and $a_\pm$ (obtained by inserting (4.29) in (4.28)) are the two roots of the equation

$$a_\pm^2 = a_\pm + 1. \tag{4.30}$$

In the limit of large M the term with the highest root dominates, and we have

$$\ln N_-(M) = (\ln a_+)\ M = \ln \frac{1+\sqrt{5}}{2}\ M. \tag{4.31}$$

By combining the information from the two estimates, we conclude that

$$\ln N(M) = d\ M. \tag{4.32}$$

where

$$\ln \frac{1+\sqrt{5}}{2} < d < \ln 2 \tag{4.33}$$

or

$$0.48 < d < 0.69. \tag{4.34}$$

Since the integers M are equally spaced, our computation yields immediately the density of microstates. The number $N(A)$ of microstates with area A grows for large A as

$$\ln N(A) = d\ \frac{A}{8\pi\hbar G} \tag{4.35}$$

This gives immediately

$$S(A) = c\,\frac{k}{\hbar G}\,A. \tag{4.36}$$

which is the Bekenstein-Hawking formula. For a different (and very elegant) derivation, see [25]. The constant of proportionality that we have obtained is

$$c = \frac{d}{8\pi}, \tag{4.37}$$

which is roughly 4π times smaller than Hawking's value $c_{Hawking} = \frac{1}{4}$.

In summary: I have argued that the black hole entropy relevant for the hole's thermodynamical interaction with its surroundings is the number of the quantum microstates of the hole which have microscopically distinct effects on the exterior of the hole. I have argued that these states are given by the quantum state of the horizon with the same area. I have counted such microstates using loop quantum gravity. I have obtained that the entropy is proportional to the area, as in the Bekenstein-Hawking formula.

Several issues remain open. I have worked in the simplified setting of a hole interacting with a given geometry, instead of working within a fully generally covariant statistical mechanics [71]. Also, it would be nice to have a direct characterization of the event horizon in the quantum theory: this could perhaps be given along the following lines. Consider a weave [4] state $|w\rangle$ which solves the hamiltonian constraint and represents a physical black hole. This can be expanded in the s-knot basis

$$|w\rangle = \sum_i c_i |s_i\rangle. \tag{4.38}$$

Consider the observables $\hat{O}_j$ representing measurement at future null infinity (for instance, see [72]). For every s_i, and all $\hat{O}_j$, define as "internal" the edges l_k of s_i such that the expectation values

$$\bar{O}_j = \langle w|\hat{O}_j|w\rangle \tag{4.39}$$

satisfy

$$\frac{d\bar{O}_j}{dl_k} = 0, \tag{4.40}$$

meaning that $\bar{O}_j$ is not affected if we change the color of l_k. A similar definition can be given for "internal vertices". Denote edges and vertices that are not internal as external. Now the quantum event horizon can be defined as the set of external edges that are nor surrounded by external edges or external vertices only. Clearly this captures the idea of the boundary between the region that "affects future null infinity" and the regions that doesn't. Notice that under this definition the quantum event horizon is just a collection of edges (pictorially: the edges cut by the horizon). This approach might clarify the issue of the type (b) and (c) intersections, and, I believe, deserves to be investigated.

Finally, the numerical discrepancy with the Hawking's value indicates that something is still poorly understood. Jacobson [73] has suggested that finite renormalization effects of the Newton constant might account for this discrepancy and has begun to explore how the presence of matter might affect it.

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