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Covariant double-null dynamics

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Abstract This is an introduction to the basics of a $(2+2)$-imbedding formalism, adapted to a double foliation of spacetime by a net of two intersecting families of lightlike hypersurfaces. It yields a simple and geometrically transparent decomposition of the Einstein equations, and has a variety of applications, e.g., to the characteristic initial-value problem (analytical and numerical), the singularity structure of Cauchy horizons and definitions of "quasi-local mass."

1 Introduction

I should like to present a brief advertisement on behalf of a $(2+2)$ lightlike decomposition of the Einstein equations recently developed by our group [1]. Formalisms of this kind becomes useful when the physics singles out particular lightlike hypersurfaces or directions, as in the characteristic initial-value problem, the dynamics of horizons, gravitational radiation, Planck-energy collisions and light-cone quantization.

A number of such formalisms have emerged over the years [2], beginning with the famous 1973 paper of Geroch, Held and Penrose [3]. Basically, all have the same content, but they *look* very different. The distinctive feature of our version is hat it is two-dimensionally covariant and thus very compact, geometrically transparent and relatively easy to use—at least, we find it so. We hope it will play a role in promoting this versatile technique (which has never really caught on with relativists) into an everyday working tool.

2 ADM $(3+1)$-decomposition: a brief recap

To set the stage and ease us into the notation, I shall briefly remind you of the elements of the familiar $(3+1)$-decomposition of Arnowitt, Deser and Misner [4].

This is based on a foliation by *spacelike* hypersurfaces t = const, with parametric equations

$$x^\alpha = x^\alpha(\xi^a, t).$$

Greek indices run from 1 to 4, and latin indices (in this section only) from 1 to 3.

The tangential base vectors $e_{(a)}$ associated with the intrinsic co-ordinates ξ^a are defined by $e^\alpha_{(a)} = \partial x^\alpha/\partial \xi^a$. The intrinsic metric and extrinsic curvature of a hypersurface t = const. are then given by

$$g_{ab} = e_{(a)} \cdot e_{(b)} \qquad K_{ab} = (\nabla_\alpha \, n_\beta) e^\alpha_{(a)} \, e^\beta_{(b)},$$

where n^α is the unit timelike normal:

$$n \cdot n \equiv g_{\alpha\beta} \, n^\alpha n^\beta = -1.$$

The 4-vector $\partial x^\alpha/\partial t$ can be decomposed into tangential and normal parts;

$$\partial x^\alpha/\partial t = s^a e^\alpha_{(a)} + N n^\alpha$$

thus defining the lapse function N and the shift vectors s^a (more conventionally written as N^a).

It follows that an arbitrary four-dimensional displacement dx^α decomposes as

$$dx^\alpha = e^\alpha_{(a)}(d\xi^a + s^a \, dt) + N n^\alpha \, dt$$

and the 4-metric as

$$ds^2 = g_{\alpha\beta} \, dx^\alpha \, dx^\beta = g_{ab}(d\xi^a + s^a \, dt)(d\xi^b + s^b \, dt) - N^2 \, dt^2.$$

The standard ADM formulae [4] now express the four-dimensional Einstein tensor components $G_{\alpha\beta}$ in terms of the intrinsic geometry (g_{ab}, ${}^{(3)}R_{ab}$) of the hypersurfaces, the extrinsic curvature K_{ab} and its normal derivative, and the lapse and shift.

However, this formalism folds in the limit where the hypersurfaces become lightlike. Because n^α is now a tangential vector, K_{ab} no longer provides extrinsic information, and the intrinsic metric g_{ab} becomes degenerate.

To deal with the lightlike case, one must fall back to a foliation of co-dimension two. I shall next sketch briefly how this works.

3 (2 + 2) lightlike decomposition: basic metric notions

We consider a double foliation of spacetime by a net of two intersecting families of lightlike hypersurfaces Σ^0 (with equations u^0 = const.) and Σ^1 (given by u^1 = const.), where $u^A(x^\alpha)$ $(A, B, \ldots, = 0, 1;\ \alpha, \beta, \ldots, = 1, \ldots, 4)$ are a given pair of scalar fields over spacetime, with lightlike gradients:

$$\nabla u^A \cdot \nabla u^B = e^{-\lambda} \eta^{AB},$$

and where the matrix

$$\eta^{AB} = \eta_{AB} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

will be used to raise and lower upper-case Latin indices, and $\lambda(x^\alpha)$ is a scalar function. The generators $\ell^{(A)}_\alpha$ of Σ^A are conveniently defined as

$$\ell^{(A)} = e^\lambda \nabla u^A.$$

Two hypersurfaces Σ^0 and Σ^1 intersect in a 2-surface S, with parametric equations

$$x^\alpha = x^\alpha(u^A, \theta^a) \qquad (a, b, \ldots = 2, 3)$$

where (θ^2, θ^3) are intrinsic co-ordinates of S. Both generators $\ell^{(A)}$ are orthogonal to S.

Holonomic basis vectors $e_{(a)}$ and the intrinsic metric of S may now be defined:

$$e^\alpha_{(a)} = \frac{\partial x^\alpha}{\partial \theta^a}, \qquad g_{ab} = e_{(a)} \cdot e_{(b)}.$$

The matrix g_{ab} and its inverse g^{ab} are used to lower and raise lower-case Latin indices, so that $e^{(a)} = g^{ab} e_{(b)}$ are the dual basis vectors tangent to S.

Two-dimensional shift vectors s^a_A are defined by

$$s^a_A = \frac{\partial x^\alpha}{\partial u^A} e^{(a)}_\alpha = -\ell^\alpha_{(A)} \frac{\partial \theta^a}{\partial x^\alpha}.$$

As in the Arnowitt-Deser-Misner formalism, the shift vector s^a_A measures how much one has to deviate from the normal direction $\ell_{(A)}$ to connect points on different 2-surfaces having the same intrinsic co-ordinates θ^a. An infinitesimal four-dimensional displacement dx^α can be decomposed as

$$dx^\alpha = \ell^\alpha_{(A)} du^A + e^\alpha_{(a)} (d\theta^a + s^a_A du^A).$$

Together with the completeness relation

$$g_{\alpha\beta} = e^{-\lambda} \eta_{AB} \ell^{(A)}_\alpha \ell^{(B)}_\beta + g_{ab} e^{(a)}_\alpha e^{(b)}_\beta$$

for the basis $(\ell^{(A)}, e_{(a)})$, this implies that the spacetime metric is decomposable as

$$g_{\alpha\beta}dx^{\alpha}\,dx^{\beta} = e^{\lambda}\eta_{AB}du^{A}\,du^{B} + g_{ab}(d\theta^{a} + s^{a}_{A}du^{A})(d\theta^{b} + s^{b}_{B}\,du^{B}).$$

4 Covariant geometrical objects embodying first derivatives of metric

Associated with its two normals $\ell_{(A)}$, a 2-surface S has two extrinsic curvatures defined by

$$K_{Aab} = (\nabla_{\beta}\ell_{(A)\alpha})e^{\alpha}_{(a)}\,e^{\beta}_{(b)}$$

and easily shown to be symmetric in a, b. (Since we are free to rescale the null vectors $\ell_{(A)}$, a certain scale-arbitrariness is inherent in this definition.)

A further basic geometrical property of the double foliation is given by the Lie bracket of $\ell_{(0)}$ and $\ell_{(1)}$. One finds

$$[\ell_{(B)}, \ell_{(A)}] = \epsilon_{AB}\omega^{a}e_{(a)} \tag{4.1}$$

where

$$\omega^{a} = \epsilon^{AB}(\partial_{B}s^{a}_{A} - s^{b}_{B}s^{a}_{A;b}).$$

The semi-colon indicates two-dimensional covariant differentiation associated with metric g_{ab}, and ϵ_{AB} is the two-dimensional permutation symbol.

The geometrical significance of the "twist" ω^{a} can be read off from (4.1): the curves tangent to the generators $\ell_{(0)}$, $\ell_{(1)}$ mesh together to form 2-surfaces (orthogonal to the surfaces S) if and only if $\omega^{a} = 0$. In this case, it would be consistent to allow the coordinates θ^{a} to be dragged along both sets of generators, and thus to gauge both shift vectors to zero.

I shall denote by D_A the two-dimensionally invariant operator associated with differentiation along the normal direction $\ell_{(A)}$. Acting on any two-dimensional geometrical object $X^{a\cdots}_{b\cdots}$, D_A is formally defined by

$$D_{A}X^{a\cdots}_{b\cdots} = (\partial_{A} - \mathcal{L}_{s^{d}_{A}})X^{a\cdots}_{b\cdots}.$$

Here, ∂_A is the partial derivative with respect to u^A and $\mathcal{L}_{s^{d}_{A}}$ the Lie derivative with respect to the 2-vector s^{d}_{A}. As an example:

$$D_{A}g_{ab} = \partial_{A}g_{ab} - 2s_{A(a;b)} = 2K_{Aab}.$$

Geometrically, $D_{A}X^{a\cdots}_{b\cdots}$ is the projection onto S of the Lie derivative with respect to $\ell_{(A)}$ of the equivalent tangential 4-tensor $X^{\alpha\cdots}_{\beta\cdots}$.

The objects K_{Aab}, ω^a and D_A are all simple projections onto S of four-dimensional geometrical objects. Consequently, they transform very simply under two-dimensional co-ordinate transformations. Under the arbitrary reparametrization

$$\theta^a \to \theta^{a'} = f^a(\theta^b, u^A) \tag{4.2}$$

(which leaves u^A and hence the surfaces Σ^A and S unchanged), ω_a and K_{Aab} transform cogrediently with

$$e_{(a)} \to e'_{(a)} = e_{(b)}\partial\theta^b/\partial\theta^{a'}.$$

By contrast, the shift vectors s^a_A undergo a more complicated gauge-like transformation, arising from the u-dependence in (4.2).

5 Ricci tensor

This geometrical groundwork is already sufficient to allow me to display the simple form that the Ricci components take in this formalism. (Notation for the tetrad components is typified by $R_{aA} = R_{\alpha\beta}e^\alpha_{(a)}\ell^\beta_{(A)}$.)

The results are

$$\begin{aligned}
{}^{(4)}R_{ab} &= \frac{1}{2}\,{}^{(2)}R g_{ab} - e^{-\lambda}(D_A + K_A)K^A_{ab} \\
&\quad + 2e^{-\lambda}K_{A(a}{}^{d}K^A_{b)d} - \frac{1}{2}e^{-2\lambda}\omega_a\omega_b - \lambda_{;ab} - \frac{1}{2}\lambda_{,a}\lambda_{,b} \\
R_{AB} &= -D_{(A}K_{B)} - K_{Aab}K_B^{ab} + K_{(A}D_{B)}\lambda \\
&\quad - \frac{1}{2}\eta_{AB}\left[(D^E + K^E)D_E\lambda - e^{-\lambda}\omega^a\omega_a + (e^\lambda)^{;a}{}_a\right] \\
R_{Aa} &= K^b_{Aa;b} - \partial_a K_A - \frac{1}{2}\partial_a D_A\lambda + \frac{1}{2}K_A\partial_a\lambda \\
&\quad + \frac{1}{2}\epsilon_{AB}e^{-\lambda}\left[(D^B + K^B)\omega_a - \omega_a D^B\lambda\right],
\end{aligned}$$

where ${}^{(2)}R$ is the curvature scalar associated with the 2-metric g_{ab}, and $K_A \equiv K^a_{Aa}$.

The economy and geometrical transparency of these formulae are self-evident. In particular, the shift vectors, which are largely an artefact of the choice of co-ordinates θ^a, make no explicit appearance.

6 Concluding remarks. Heuristics

This formalism has so far been applied to the analysis of the characteristic initial-value problem [1] and the nature of the infinite-blueshift singularity along the Cauchy horizon of a spinning black hole [5]. I expect other applications, both analytic and numerical, to follow.

I should like to conclude with some heuristic remarks. Spherically symmetric space-times are naturally and very simply described by a $(2+2)$ formalism [6]. It is remarkable that the generic formulae can often be cast in a form that resemble the spherical ones, with simple and physically intuitive modifications to allow for the presence of gravitational radiation [7].

As a sample, let me recall that any spherisymmetric metric can be expressed as [6]

$$ds^2 = g_{AB} dx^A\, dx^B + r^2(x^A)\, d\Omega^2$$

where x^A $(A = 0, 1)$ are arbitrary co-ordinates for the quotient space M^4/S^2. The usual Schwarzschild mass function $M(x^A)$ is defined by [6]

$$1 - 2M(x^A)/r = g^{AB}(\partial_A r)(\partial_B r).$$

Then it follows from the Einstein equations with stress-energy tensor $T^{\alpha\beta}$ that M satisfies a $(1+1)$-dimensional wave equation [6]

$$\Box M \equiv g^{AB}\nabla_A \nabla_B M = -16\pi^2 r^3 T_{AB} T^{AB} + \cdots$$

where the dots represent terms linear in $T_{\alpha\beta}$, which are relatively small in regions of large blueshift, e.g., near a Cauchy horizon. This remarkable formula brings out explicitly the effects of the nonlinearity of the Einstein equations.

There is a generic counterpast of this equation in double-null dynamics [7]. Let us define a generic "Schwarzschild mass function" $M(x^\alpha)$ by

$$1 - 2M/r = e^{-\lambda}\eta^{AB}(D_A r)(D_B r)$$

where the area-function $r(x^\alpha)$ is defined by

$$({}^{(2)}g)^{\frac{1}{2}} = C(\theta^a) r^2$$

(C is arbitrary and can be fixed by an initial condition). Then

$$D^A D_A M = -16\pi^2 r^3 (T_{AB} + \tau_{AB})(T^{AB} + \tau^{AB}) + \cdots$$

where the gravitational-wave stress-energy is defined by

$$\tau_{AB} = \frac{1}{8\pi}\left(\sigma_{Aab}\sigma_B^{ab} - \frac{1}{2}\eta_{AB}\sigma_{Dab}\sigma^{Dab}\right)$$

in terms of the shear $\sigma_{Aa}{}^{b} = K_{Aa}{}^{b} - \frac{1}{2}\delta_a{}^b K_A$.

Of course, these are merely words. No operational prescription exists for defining notions like "quasi-local mass" and stress-energy of gravitational waves, except in certain limiting cases (very high frequencies). Words can nevertheless be quite useful as a heuristic guide to complex formal calculations, especially so in situations where the "fluff" represented by the dots is relatively small.

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