# Parasupersymmetric quantum mechanics with generalized deformed parafermions

Autor(en): **Beckers, Jules / Debergh, Nathalie / Quesne, Christiane**

# Parasupersymmetric Quantum Mechanics with Generalized Deformed Parafermions

By  Jules Beckers [1], Nathalie Debergh [2]

Theoretical and Mathematical Physics, Institute of Physics (B.5), University of Liège, B-4000 Liège 1, Belgium

and Christiane Quesne [3]

Physique Nucléaire Théorique et Physique Mathématique, Université Libre de Bruxelles, Campus de la Plaine CP229, Boulevard du Triomphe, B-1050 Brussels, Belgium

*Abstract.*  A superposition of bosons and generalized deformed parafermions corresponding to an arbitrary paraquantization order $p$ is considered to provide deformations of parasupersymmetric quantum mechanics. New families of parasupersymmetric Hamiltonians are constructed in connection with two examples of su(2) nonlinear deformations such as introduced by Polychronakos and Roček.

# 1   Introduction

During the last few years, nonlinear deformations of (the universal enveloping algebra of) Lie algebras have attracted a lot of attention (see Ref. [1], and references therein). They include some specific deformations with a Hopf algebraic structure, often called $q$-algebras

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[1] E-mail: beckers@vm1.ulg.ac.be

[2] Chercheur, Institut Interuniversitaire des Sciences Nucléaires

[3] Directeur de recherches, Fonds National de la Recherche Scientifique; E-mail: cquesne@ulb.ac.be

and related to quantum groups [2], as well as more general deformations, such as those of su(2) introduced by Polychronakos [3] and Roček [4].

Biedenharn [5] and Macfarlane [6] pioneering works on the $q$-deformed harmonic oscillator have been extended by various authors. Besides a specific attempt [7], the introduction of the generalized deformed oscillator [8] has proved useful to provide a unified description [9] of the Bose, Fermi, parabose and parafermi harmonic oscillators [10], as well as their $q$-deformations [11].

Quite recently [12], generalized deformed parafermions, including the $q$-deformed ones as a special case, were defined in the generalized deformed oscillator framework, and were shown to be related to some unitary irreducible representations (unirreps) of the Polychronakos and Roček su(2) deformations. Moreover, some physically relevant exactly solvable Hamiltonians, such as the Morse and modified Pöschl-Teller ones, were proved to be equivalent to Fermi-like oscillator Hamiltonians constructed in terms of these generalized deformed parafermions, which therefore provide a new algebraic description of their bound state spectrum [12].

In the present paper, we will consider a superposition of standard bosons and such generalized deformed parafermions of arbitrary paraquantization order $p$ to study deformations of parasupersymmetric quantum mechanics, as initiated in a previous work [13]. Our approach will differ with respect to that using bosons and $q$-deformed parafermions of order $p$, developed in a recent paper [14].

Our purpose will be twofold: firstly, to introduce generalized deformations of the parasuperalgebra Psqm(2), generated by two (parasuper)charges $Q$ and $Q^\dagger$, and a parasupersymmetric Hamiltonian $H$, and to establish a general (necessary and sufficient) condition for the existence of new (nontrivial) types of Hamiltonians; secondly, to work out in some detail two examples, connected with two Polychronakos and Roček algebras [3], [4] already considered before [12], in order to point out how some new (nontrivial) properties and results can be obtained.

# 2  Generalized Deformations of the Parasuperalgebra Psqm(2)

Let $b$ and $b^\dagger$ denote generalized deformed parafermionic operators, as defined in Proposition 2 of Ref. [12], i.e., operators satisfying the nilpotency relations

$$b^{p+1} = 0 \qquad (b^\dagger)^{p+1} = 0 \tag{2.1}$$

and the trilinear relations

$$[b, [b^\dagger, b]] = G(N)b \qquad [b^\dagger, [b, b^\dagger]] = b^\dagger G(N) \tag{2.2}$$

where

$$G(N) = 2F(N+1) - F(N) - F(N+2) \qquad F(N) = b^\dagger b \tag{2.3}$$

$F$ being any positive analytic function [4].

Let us recall that for standard parafermions

$$G(N) = 2 \qquad F(N) = F(p+1-N) = N(p+1-N) \qquad (2.4)$$

so that relations (2.1)–(2.3) reduce to the original characterization of parastatistics [10]. In general, we shall replace the factor $k(p+1-k)$ by the function $F(p+1-k)$, so that the matrix realization of the generalized deformed parafermionic operators (see Ref. [14]) is given by

$$b = \sum_{k=1}^{p} [F(p+1-k)]^{1/2} e_{k+1,k} \qquad b^\dagger = \sum_{k=1}^{p} [F(p+1-k)]^{1/2} e_{k,k+1} \qquad (2.5)$$

in terms of $(p+1)$-dimensional matrices $e_{m,n}$ with entry 1 at the intersection of row $m$ and column $n$ and zeroes everywhere else.

Let us modify the generalized parasupercharges $Q$ and $Q^\dagger$ introduced in the $q$-deformations of Psqm(2) study [14], by substituting the operators (2.5) for the $q$-deformed parafermionic operators. Hence, $Q$ and $Q^\dagger$ are now written as

$$Q = \sum_{k=1}^{p} \left[ \tfrac{1}{2} F(p+1-k) \right]^{1/2} (p_x + iW_k(x)) \, e_{k+1,k}$$

$$Q^\dagger = \sum_{k=1}^{p} \left[ \tfrac{1}{2} F(p+1-k) \right]^{1/2} (p_x - iW_k(x)) \, e_{k,k+1} \qquad (2.6)$$

where $p_x = -id/dx$, and $W_k(x)$, $k = 1, 2, \ldots, p$, refer to the parasuperpotentials inside the bosonic operators. In correspondence with eqs. (2.1)–(2.3), such charges have to satisfy the nilpotency relations

$$Q^{p+1} = 0 \qquad (Q^\dagger)^{p+1} = 0 \qquad (2.7)$$

as well as the structure ones

$$[Q, [Q^\dagger, Q]] = G(N)QH \qquad [Q^\dagger, [Q, Q^\dagger]] = Q^\dagger H G(N) \qquad (2.8)$$

where $H$ plays the role of the deformed parasupersymmetric Hamiltonian, with respect to which the charges are conserved, i.e.,

$$[H, Q] = 0 \qquad [H, Q^\dagger] = 0. \qquad (2.9)$$

Let us point out that relations (2.7)–(2.9) characterize the deformed parasupersymmetric algebra associated with our superposition of bosons and generalized deformed parafermions.

As in the $q$-deformed case [14], the structure relations (2.8) imply some constraints on the superpotentials in the form of Riccati equations

$$W_{k+1}^2 + W_{k+1}' = W_k^2 - W_k' + c_k \qquad k = 1, 2, \ldots, p-1 \qquad (2.10)$$

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[4]In Ref. [12], $F$ was assumed to be strictly positive on the set $\{1, 2, \ldots, p\}$. Here, we shall only assume that it is nonnegative.

where primes refer to space derivatives, and the $c_k$'s are so far arbitrary constants. The latter will be chosen by requiring that the parasupersymmetric Hamiltonian is diagonal with non-vanishing matrix elements given by

$$H_{k,k} = \tfrac{1}{2}p_x^2 + f_k(x) \qquad k = 1, 2, \ldots, p+1 \tag{2.11}$$

in terms of some functions $f_k(x)$ to be determined.

By introducing eq. (2.6) into the two trilinear relations (2.8), we obtain after some relatively tedious calculations two sets of $p$ equations, given by

$$\begin{aligned}
G(p-k)H_{k,k} &= \tfrac{1}{2}G(p-k)(p_x^2 + W_k^2 + W_k') + \tfrac{1}{2}[F(p+2-k)c_{k-1} - F(p-k)c_k] \\
&\quad k = 1, 2, \ldots, p
\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}
G(p+1-k)H_{k,k} &= \tfrac{1}{2}G(p+1-k)(p_x^2 + W_k^2 + W_k') + \tfrac{1}{2}[F(p+3-k)(c_{k-2} - c_{k-1}) \\
&\quad -2F(p+2-k)c_{k-1}] \qquad k = 2, 3, \ldots, p \\
G(0)H_{p+1,p+1} &= \tfrac{1}{2}G(0)(p_x^2 + W_p^2 - W_p') + \tfrac{1}{2}F(2)c_{p-1}
\end{aligned} \tag{2.13}$$

respectively. The latter lead to two different expressions for $H_{k,k}$, $k = 2, 3, \ldots, p$. By equating them, we get a system of $p-1$ homogeneous linear equations in $p-1$ unknowns $c_k$, $k = 1, 2, \ldots, p-1$.

Such a system always admits the trivial solution $c_k = 0$, $k = 1, 2, \ldots, p-1$. For the corresponding parasupersymmetric Hamiltonian, we then recover the original result of undeformed Psqm(2) for an arbitrary paraquantization order [14]. The system however also admits a nontrivial solution (i.e., with at least one nonzero arbitrary constant), absent in the undeformed case, if and only if the determinant of its coefficients vanishes. We did establish the general form of this necessary and sufficient condition. For brevity's sake, we only quote here the final result:

$$(p+1)F(1)F(2)\ldots F(p)G(1)G(2)\ldots G(p-2) = 0. \tag{2.14}$$

We conclude that a nontrivial solution does exist if and only if one of the arbitrary functions $F(k)$ or $G(k)$ vanishes for some $k \in \{1, 2, \ldots, p\}$ or $\{1, 2, \ldots, p-2\}$, respectively. The diagonal elements of the corresponding parasupersymmetric Hamiltonian are then given by either equation (2.12) or (2.13).

In the next section, we will show on two examples that nontrivial solutions do indeed exist.

# 3   Examples

The examples to be considered here correspond to generalized deformed parafermionic operators transforming under a $(p + 1)$-dimensional unirrep of some Polychronakos [3] and

Roček [4] deformed su(2) algebra (instead of su(2), as in the undeformed case). Such a nonlinear algebra is defined by the commutation relations

$$[J_0, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = f(J_0) \tag{3.1}$$

where $f(J_0)$ is some real, analytic function in $J_0$, going to $2J_0$ for some limiting values of the parameters. Whenever $f(J_0)$ is a polynomial of degree less than or equal to three, the functions $F(N)$ and $G(N)$ of eqs. (2.2) and (2.3), characterizing the generalized deformed parafermions, are given by [5]

$$
\begin{aligned}
F(N) &= N(p+1-N)(\lambda + \mu N + \nu N^2) \\
G(N) &= 2\left\{\lambda - (p-2)\mu - (3p-4)\nu + 3[\mu - (p-3)\nu]N + 6\nu N^2\right\}
\end{aligned}
\tag{3.2}
$$

where the constants $\lambda$, $\mu$, $\nu$ can be found from $f(J_0)$ as explained in Proposition 5 of Ref. [12].

The first example [4], [12] corresponds to

$$f(J_0) = 2J_0 + \alpha J_0^2 \tag{3.3}$$

where

$$|\alpha| < \frac{6}{2p+1} \tag{3.4}$$

ensures the existence of a $(p+1)$-dimensional unirrep characterized by a highest weight $j = \frac{1}{2}p - \alpha^{-1}(1-\epsilon)$, where $\epsilon = \left[1 - \frac{1}{12}\alpha^2 p(p+2)\right]^{1/2}$. The functions $F(N)$ and $G(N)$ are then given by (3.2), where $\lambda = -\frac{1}{6}\alpha(p+1) + \epsilon$, $\mu = \frac{1}{3}\alpha$, and $\nu = 0$.

As $F(k) > 0$ for $k = 1, 2, \ldots, p$, since the representation considered is irreducible and unitary, condition (2.14) can only be satisfied provided $G(k)$ vanishes, i.e.,

$$\epsilon = \frac{1}{2}(p - 1 - 2k)\,\alpha \tag{3.5}$$

for some $k \in \{1, 2, \ldots, p-2\}$. Condition (3.5) is equivalent to

$$\alpha = 2\sqrt{3}\,\sigma \left[4p^2 - 4(3k+1)p + 3(2k+1)^2\right]^{-1/2} \tag{3.6}$$

where $\sigma$ denotes the sign of $\frac{1}{2}(p-1) - k$. The inequality (3.4) here requires

$$(2p - 3k - 2)(p - 3k - 1) > 0 \tag{3.7}$$

which is possible only for $p \geq 5$. For an arbitrary paraquantization order $p = 3l - 1$, $3l$, or $3l + 1$, where $l \geq 2$, the allowed parameter values leading to new parasupersymmetric Hamiltonians are therefore given by (3.6), where $k = 1, 2, \ldots, l - 1$ or $k = p - l, p - l + 1, \ldots, p - 2$.

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[5]In eq. (3.2), some misprints in Ref. [12] have been corrected.

The minimal context $p = 5$ only allows the values $k = 1$ and $k = 3$, leading to specific parameter values $\alpha = 2\sqrt{3/47}$, and $\alpha = -2\sqrt{3/47}$, respectively. In such a case, we therefore obtain two new families of parasupersymmetric Hamiltonians, whose diagonal matrix elements are given by

$$
\begin{aligned}
H_{i,i} &= \tfrac{1}{2}(p_x^2 + W_i^2 + W_i') + \gamma_i & i = 1, 2, \ldots, 5 \\
H_{6,6} &= \tfrac{1}{2}(p_x^2 + W_5^2 - W_5') + \gamma_5
\end{aligned} \tag{3.8}
$$

where

$$
\gamma_1 = -\tfrac{8}{9}c_1 \qquad \gamma_2 = -\tfrac{25}{18}c_1 \qquad \gamma_3 = -\tfrac{200}{81}c_1 \qquad \gamma_4 = -\tfrac{50}{9}c_1 \qquad \gamma_5 = -\tfrac{400}{9}c_1 \tag{3.9}
$$

and

$$
\gamma_1 = \tfrac{4}{3}c_1 \qquad \gamma_2 = \tfrac{5}{6}c_1 \qquad \gamma_3 = \tfrac{2}{27}c_1 \qquad \gamma_4 = \tfrac{1}{24}c_1 \qquad \gamma_5 = \tfrac{2}{75}c_1 \tag{3.10}
$$

respectively.

Let us also notice that besides these general results, two other specific cases have to be distinguished. They are associated with the possible vanishing of either $G(0) = 2F(1) - F(0)$ or $G(p-1) = 2F(p) - F(p-1)$, and correspond to the parameter values $\alpha = 2\sqrt{3}(4p^2 - 4p + 3)^{-1/2}$, or $\alpha = -2\sqrt{3}(4p^2 - 4p + 3)^{-1/2}$. Such values satisfy the inequality (3.4), but leave the matrix element $H_{p+1,p+1}$ or $H_{1,1}$ entirely arbitrary, thereby excluding the knowledge of the corresponding parasupersymmetric Hamiltonian final form.

The second example [12] corresponds to

$$
f(J_0) = 2J_0 + \alpha J_0^3 \tag{3.11}
$$

where

$$
\alpha > -\frac{8}{p^2} \tag{3.12}
$$

ensures the existence of a $(p + 1)$-dimensional unirrep characterized by $j = p/2$. The corresponding functions $F(N)$ and $G(N)$ are given by (3.2), where $\lambda = 1 + \tfrac{1}{8}\alpha p(p + 2)$, $\mu = -\tfrac{1}{4}\alpha(p + 1)$, and $\nu = \tfrac{1}{4}\alpha$.

Once again, condition (2.14) will be satisfied provided $G(k) = 0$, i.e.,

$$
\alpha = -8[3p^2 - 6(2k + 1)p + 12k^2 + 12k + 4]^{-1} \tag{3.13}
$$

for some $k \in \{1, 2, \ldots, p - 2\}$. Such a parameter value satisfies the inequality (3.12) if

$$
p < \tfrac{3}{2}(2k + 1) - \tfrac{1}{2}(12k^2 + 12k + 1)^{1/2} \quad \text{or} \quad p > \tfrac{3}{2}(2k + 1) + \tfrac{1}{2}(12k^2 + 12k + 1)^{1/2}. \tag{3.14}
$$

A detailed discussion of these conditions leads to the result that the allowed parameter values giving rise to new parasupersymmetric Hamiltonians correspond to $k = 1, 2, \ldots, l - 1$, or $k = p - l, p - l + 1, \ldots, p - 2$, and

$$
4l + \kappa_l \leq p < 4(l + 1) + \kappa_{l+1} \tag{3.15}
$$

where $\kappa_l$ is the integer determined by the condition

$$-l - \tfrac{3}{2} + \tfrac{1}{2}(12l^2 - 12l + 1)^{1/2} < \kappa_l \le -l - \tfrac{1}{2} + \tfrac{1}{2}(12l^2 - 12l + 1)^{1/2}. \qquad (3.16)$$

We also observe that the matrix elements $H_{1,1}$ and $H_{p+1,p+1}$ are left arbitrary if $p > 2$ and $\alpha = -8(3p^2 - 6p + 4)^{-1}$.

The lowest paraquantization order for which conditions (3.15) and (3.16) are satisfied is $p = 8$. Then $G(1) = G(6) = 0$ for $\alpha = -2/19$. The corresponding family of new parasupersymmetric Hamiltonians is defined by the following set of diagonal matrix elements:

$$\begin{aligned}
H_{i,i} &= \tfrac{1}{2}(p_x^2 + W_i^2 + W_i') + \gamma_i \qquad i = 1, 2, \ldots, 8 \\
H_{9,9} &= \tfrac{1}{2}(p_x^2 + W_8^2 - W_8') + \gamma_8
\end{aligned} \qquad (3.17)$$

where

$$\begin{aligned}
\gamma_1 &= \tfrac{7}{6}c_1 \qquad \gamma_2 = \tfrac{2}{3}c_1 \qquad \gamma_3 = \tfrac{7}{24}c_1 - \tfrac{15}{4}c_3 \qquad \gamma_4 = \tfrac{7}{24}c_1 - \tfrac{17}{4}c_3 \\
\gamma_5 &= \tfrac{7}{20}c_1 - \tfrac{11}{2}c_3 \qquad \gamma_6 = \tfrac{49}{96}c_1 - \tfrac{135}{16}c_3 \qquad \gamma_7 = c_1 - \tfrac{120}{7}c_3 \\
\gamma_8 &= \tfrac{329}{48}c_1 - \tfrac{975}{8}c_3.
\end{aligned} \qquad (3.18)$$

# 4  Some Comments

In conclusion, we did show through two examples that the approach developed in the present paper leads to families of new deformed parasupersymmetric Hamiltonians. As compared with those generated by the previous approach based upon $q$-deformed parafermions [14], the latter correspond to rather high paraquantization orders.

As a last comment, we would like to mention that the $p = 2$ case can be discussed in full details. The corresponding system obtained from (2.12) and (2.13), as well as condition (2.14) lead to the following three Hamiltonians

$$H^{(1)} = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}\begin{pmatrix} W_1^2 + W_1' & 0 & 0 \\ 0 & W_2^2 + W_2' - c_1 & 0 \\ 0 & 0 & W_2^2 - W_2' - c_1 \end{pmatrix} \qquad (4.1)$$

if $F(1) = 0$,

$$H^{(2)} = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}\begin{pmatrix} W_1^2 + W_1' + c_1 & 0 & 0 \\ 0 & W_2^2 + W_2' & 0 \\ 0 & 0 & W_2^2 - W_2' \end{pmatrix} \qquad (4.2)$$

if $F(2) = 0$, and

$$H^{(3)} = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}\begin{pmatrix} W_1^2 + W_1' & 0 & 0 \\ 0 & W_2^2 + W_2' & 0 \\ 0 & 0 & W_2^2 - W_2' \end{pmatrix} \qquad (4.3)$$

otherwise. Let us notice that the first two cases lead to pseudostatistical considerations [15].

For oscillator-like interactions characterized by $W_1 = \omega x$ ($\omega$ being the angular frequency), the Hamiltonians (4.1)–(4.3) become

$$H^{(1)} = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}\omega^2 x^2 + \tfrac{1}{2}\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{pmatrix} \tag{4.4}$$

$$H^{(2)} = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}\omega^2 x^2 + \tfrac{1}{2}\omega \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{4.5}$$

$$H^{(3)} = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}\omega^2 x^2 + \tfrac{1}{2}\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.6}$$

respectively. They can be interpreted as Hamiltonians of a system consisting of a non-interacting three-level subsystem and one bosonic mode, as occurring in quantum optics [16]. The third possibility (4.6) corresponds to the so-called ∨-type and is the only one available in the undeformed and $q$-deformed [5, 6] contexts. Our general deformation (2.6) completes the information by allowing the other possible scheme of three-level configurations, namely the Ξ-type.

Moreover, the Hamiltonian $H^{(1)}$ supplemented by the constant term $\tfrac{1}{2}\omega$ (or, equivalently, $H^{(2)}$ supplemented by $-\tfrac{1}{2}\omega$) has a clear physical interpretation as describing the motion of a spin-1 particle in both an oscillator potential and a homogeneous magnetic field [17]. Once again, this result is relevant to our general deformation (2.6), but is not possible either in the undeformed or the $q$-deformed context.

Whether some other examples, associated with specific sets of analytic functions $F(N)$, may be of physical interest remains an open question, to which we hope to come back in a near future.

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