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Deterministic Evolutions and Schrödinger Flows*

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Abstract. Deterministic evolutions are defined without *ad hoc* hypotheses but in the context of the axiomatic approach due to Aerts and Piron. For systems described by a family of Hilbert spaces one can prove necessary conditions on the resulting flow using the tools of projective geometry. In particular we show that a deterministic evolution is always given by a family of partially defined unitary operators.

1 Introduction

In order to find a natural definition of a concept such as deterministic evolution one should take into account the meaning of the primitive notions of the underlying theory. We will therefore analyse the structure of an evolution in terms of the New Quantum Mechanics following the approach of Aerts [1982] and Piron [1990]. We insist on the fact that our approach is realistic and is based on the usual logic. Further, we remark that if some probability appears in a calculation it is just the classical standard one.

Our aim is to capture as much as possible of the essence of an evolution by taking a complete set of Einstein elements of reality of the system. Of course we will not be able to specify all kinds of evolution, since obviously the very nature of the system may be changed in the course of an evolution. For example, if one violently accelerates an electron it will create pairs (and not even just lepton pairs).

In the following we will therefore restrict ourselves to smooth evolutions, which do not

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suffer such processes. Here one can physically define maps which tell us much about the structure of an evolution. The starting point is the following remark due to Daniel [1989]: given an evolution one can induce an associated map from the set of final properties into the set of initial ones.

2 Preliminaries

We will base our definition of the evolution on the approach to physical theories developed by Aerts and Piron. In this section we briefly review the definitions that we will need in the following. In particular we will discuss the general structure of physical theories and the special case where the state space can be canonically realised by a family of Hilbert spaces.

Let us recall the following fundamental concepts. A property is defined by an experimental project that one could realise with the system and where one has chosen in advance what would be the positive result if one were to perform the experiment. For a given well-defined particular system such a property is said to be actual for the system if it is such that *if* one were to perform the project then the positive result would be certain. Thus an actual property is nothing else than an Einstein element of reality possessed by the system. In general a given property of the system is only potential, it is actual only in some special well-defined cases and according to a given preparation of the system [Piron 1990].

On a collection $\mathcal{Q}$ of experimental projects relative to a given system we can define a partial preorder relation. An experimental project α is said stronger than another β, written $\alpha < \beta$, if in each case that the positive response would be certain for α the positive response would also be certain for β. The properties can then be identified with the equivalence classes of experimental projects and so they form a set $\mathcal{L}$ equipped with a partial order relation.

We will also need the notion of state as defined by Aerts and Piron: the state of a given well-defined particular system is the complete subset of its actual properties (here complete means complete for the realisable experimental projects). Hence, knowing the state we know everything that can be obtained from it with certainty. Naturally, since the time by its very nature is always changing, any state also changes, either by itself or under some external influence. Hence we can associate to each system the set Σ of all of its possible states.

Here we have considered states as being subsets of the set of properties. Dually, one can consider as given in the beginning the set Σ of all possible states and then represent each property by that subset comprising exactly those possible states for which the given property is actual. So, by their very definitions, the order relation on $\mathcal{L}$ then coincides with the set theoretical inclusion, and the greatest lower bound with the intersection.

In addition to this first mathematical structure, the partial order defined on $\mathcal{L}$, we will also introduce a second mathematical structure, the orthogonality relation, defined on

Σ. Following Aerts and Piron we call two states $\mathcal{E}_i$ and $\mathcal{E}_j$ orthogonal, written $\mathcal{E}_i \perp \mathcal{E}_j$, if there exists an experimental project α for which the positive result would be certain in the state $\mathcal{E}_i$ and impossible in the state $\mathcal{E}_j$ (if we were indeed to perform the experiment). Such an α will be said to separate $\mathcal{E}_i$ and $\mathcal{E}_j$ in the following.

By applying some old widely accepted physical principles one can show that the properties of the system are the biorthogonal subsets of Σ. Further, one can also prove that each singleton p is already biorthogonal, is an atom of $\mathcal{L}$, and corresponds to a well defined possible state, $\mathcal{E} = \{a \in \mathcal{L} \,|\, p < a\}$. It can then be proved that the property lattice decomposes into irreducible components which turn out to be indexed by classical variables (variables taking well defined values in each one of the possible states) [Piron 1990, §1.7].

For a wide class of physical systems one can in fact say more. These are entities, that is systems which exist as indivisible individuals [Aerts 1982]. For an entity the experimental projects act on the system as a whole: in the classical case an entity is in general nothing else that a point particle, whereas in the quantum case it is also called a particle but it is intrinsically a nonlocal object.

In such a case the system satisfies further laws (weak modularity and covering) and one can then prove that its set of possible states Σ defines a projective geometry. In this geometry a point is a state of Σ and a line defined by two different points $\mathcal{E}_i$ and $\mathcal{E}_j$ is the subset of states contained in the least upper bound $\mathcal{E}_i \vee \mathcal{E}_j$. The irreducible components of the lattice correspond exactly to the irreducible components of the projective geometry and each such component is a Hilbert geometry: the biorthogonal subsets (closed linear manifolds) of some Hilbert space [Piron 1976, §3.1].

An (almost stable) particle is defined as an entity which admits essentially just the observables position, momentum and time, each one defined by some covariance relation called an imprimitivity system. For such systems the time is always a classical variable and so the lattice $\mathcal{L}$ is just the direct union $\mathcal{L} = \bigvee_t \mathcal{L}_t$, where in the elementary cases each $\mathcal{L}_t$ can be realised in the Schrödinger representation as a copy of the lattice of biorthogonal subsets of $L^2(\mathbb{R}^3)$ or $\mathbb{C}^2 \otimes L^2(\mathbb{R}^3)$[Piron 1976, §5.1]. In the following we will in fact consider more general systems which nevertheless can also be decomposed in this way.

3 Evolutions

Obviously the possibility of choosing an evolution for the system generates *a priori* new possible experimental projects. Indeed a given evolution is nothing more than a part of an experimental project. For the internal consistency of the theory, and in fact from the very definition of the state, these new projects must already have their equivalent in the collection of all possible experimental projects. This remark allows us to derive conditions on the physically imposable evolutions of a system in the following way. Given an experimental project α_1 in $\mathcal{Q}_{t_1}$ for some future time t_1, the given evolution defines at the present time t_0 a new experimental project α_0 in $\mathcal{Q}_{t_0}$ by the prescription "evolve the system as required from time t_0 to time t_1 and perform α_1" [Daniel 1989]. We write

$\alpha_0 = \Phi_{01}\alpha_1$.

According to the usual ideas of causality in time, we will make the hypothesis that during the evolution the system cannot anticipate the experimental project which will be performed at time t_1. With such a hypothesis we can deduce the following:

Lemma 3.1 *Φ_{01} induces a map $\phi_{01} : \mathcal{L}_{t_1} \to \mathcal{L}_{t_0}$ which preserves the order and greatest lower bound, and maps O_1 to O_0.*

Proof: By hypothesis, the positive response for $\Phi_{01}\alpha_1$ is certain at time t_0 if and only if the final state obtained at time t_1 by the evolution is such that α_1 is certain. This means that the property defined by $\Phi_{01}\alpha_1$ does not depend on the choice in the equivalence class of α_1. In other words Φ_{01} defines a map $\phi_{01} : \mathcal{L}_{t_1} \to \mathcal{L}_{t_0}$ which preserves the order given by the set theoretical inclusion and the greatest lower bound given by the intersection. Finally it is clear that O_1 is mapped to O_0. ∎

The existence of such a map translates the fact that the given evolution must satisfy certain conditions. In fact $a_0 < \phi_{01}(a_1)$ means that if the property a_0 was actual at time t_0 and if we were to perform the evolution, then the property a_1 would be actual at time t_1. Of course for $a_0 < \phi_{01}(a_1)$ it is necessary that $a_0 < \phi_{01}(I_1)$, since ϕ_{01} preserves the order. Recall that I_1 is the maximal property of the lattice, which asserts the existence of the system at time t_1. Indeed not all states defined at time t_0 would produce a final state since the system can very well disappear. For example, if we describe an atom in a superconducting cavity at low temperature and high quality factor, the future state of the system will only exist if the atom has stayed in the cavity.

Since $\phi_{01} : \mathcal{L}_{t_1} \to \mathcal{L}_{t_0}$ preserves the greatest lower bound we can define a map $\psi_{10} : [O_0, \phi_{01}(I_1)] \to \mathcal{L}_{t_1}$, which is called the Galois dual since $a_0 < \phi_{01}(a_1)$ implies $\psi_{10}(a_0) < a_1$ and the converse.

Theorem 3.2 *Let us define $\psi_{10} : [O_0, \phi_{01}(I_1)] \to \mathcal{L}_{t_1}$ by $\psi_{10}(a_0) = \bigwedge\{a_1 \mid a_0 < \phi_{01}(a_1)\}$. Then ψ_{10} is the Galois dual of ϕ_{01} and so preserves the order and least upper bound, and maps O_0 to O_1.*

Proof: We show that ψ_{10} is a Galois dual. By the very definition of ψ_{10}, $a_0 < \phi_{01}(a_1)$ implies that $\psi_{10}(a_0) < a_1$. For the converse we first remark that, since $a_0 < \phi_{01}(I_1)$,

$$a_0 < \bigwedge\{\phi_{01}(a_1) \mid a_0 < \phi_{01}(a_1)\} = \phi_{01}\psi_{10}(a_0).$$

Then if $\psi_{10}(a_0) < a_1$ we have that $a_0 < \phi_{01}\psi_{10}(a_0) < \phi_{01}(a_1)$ since ϕ_{01} preserves the order.

Next, we first show that ψ_{10} preserves the order. Let $a_0 < b_0$. Then

$$\psi_{10}(a_0) = \bigwedge\{x_1 | a_0 < \phi_{01}(x_1)\} < \bigwedge\{x_1 | b_0 < \phi_{01}(x_1)\} = \psi_{10}(b_0).$$

Second ψ_{10} then also preserves the greatest lower bound. Indeed let us consider a family of properties $\{a_{1,\alpha}\}$ indexed by α. Since ψ_{10} preserves the order, $\psi_{10}\big(\vee_\alpha a_{1,\alpha}\big)$ is an upper bound, and it is the least upper bound. Indeed, if b_1 is an upper bound, $\psi_{10}(a_{1,\alpha}) < b_1$ for all $a_{1,\alpha}$, then $a_{1,\alpha} < \phi_{01}(b_1)$ by Galois duality and so $\vee_\alpha a_{1,\alpha} < \phi_{01}(b_1)$. Hence $\psi_{10}\big(\vee_\alpha a_{1,\alpha}\big) < b_1$ also by Galois duality. Finally $\psi_{10}(O_0) = \bigwedge\{x_1 | O_0 < \phi_{01}(x_1)\} = O_1$ since $\phi_{01}(O_1) = O_0$.

For the sake of completeness we also show the unicity of the Galois dual. We have just proved that $a_0 < \phi_{01}\psi_{10}(a_0)$. If χ_{10} is also a Galois dual then this implies that $\chi_{10}(a_0) < \psi_{10}(a_0)$ and by interchanging the roles of ψ_{10} and χ_{10} we have also $\psi_{10}(a_0) < \chi_{10}(a_0)$. ■

The property $\psi_{10}(p_0)$ is the strongest property which is actual by the evolution for the initial state defined by the atom p_0. If this property is also an atom, then the final state is completely defined and in this situation we will call the motion deterministic. If the motion is deterministic for each of the initial states contained in $\phi_{01}(I_1)$ we will call the evolution maximal deterministic, since for the others the final state is not even defined, the system having the possibility to disappear.

This is exactly the situation which occurs in the classical Kepler problem: some initial states lead to a collision with the sun, and this collision is not deterministic and not described within Newtonian theory alone. Another simple example is given by a measurement made with a Stern-Gerlach apparatus. To realise such an apparatus we put an absorbing screen in one of the outgoing beams, and it is only for some initial states that the particle will certainly not be absorbed. By definition, such initial states are in $\phi_{01}(I_1)$, and only for these states is the evolution deterministic.

From the philosophical point of view, we must suppose strong determinism and so a maximal deterministic evolution. However, to be able to predict exactly the final state we must know completely the initial state, a situation which is in fact almost never realisable in practice. This all the more since to define precisely the state one must also define its classical variables, and these can be chaotic.

4 Maximal Deterministic Evolutions of Entities

In this section we will restrict our attention to the particular case of an entity which is described by a family of complex Hilbert spaces indexed by the time. As discussed in section 2, in this case the set Σ_{t_0} of atoms of $\mathcal{L}_{t_0}$ and the set Σ_{t_1} of atoms of $\mathcal{L}_{t_1}$ each define projective geometries. According to the above philosophy we suppose that the evolution is maximally deterministic. The evolution then defines a map ψ_{10} from the projective geometry $\tilde{\Sigma}_{t_0}$ built with the subsets of states in $\phi_{01}(I_1)$ to that built with Σ_{t_1}.

This map is then a particular case (in fact with empty kernel) of a morphism of projective geometries as recently defined by Faure and Frölicher [1993]. Indeed let three atoms be such that $p_0 < q_0 \vee r_0 < \phi_{01}(I_1)$, then $\psi_{10}(p_0) < \psi_{10}(q_0 \vee r_0) = \psi_{10}(q_0) \vee \psi_{10}(r_0)$.

The morphism ψ_{10} decomposes (the irreducible equivalence classes of states are mapped into irreducible equivalence classes). We recall that two distinct points p and q are called equivalent if there exists a third point r on the line defined by p and q. Now let $p_0, q_0 < \phi_{01}(I_1)$ be distinct and equivalent so that there exists $r_0 < p_0 \vee q_0$. Then $\psi_{10}(r_0) < \psi_{10}(p_0) \vee \psi_{10}(q_0)$. If $\psi_{10}(p_0) = \psi_{10}(q_0)$ then the two are clearly equivalent. If $\psi_{10}(p_0) \neq \psi_{10}(q_0)$ then $\psi_{10}(p_0) < \psi_{10}(r_0) \vee \psi_{10}(q_0)$ so that $\psi_{10}(r_0) \neq \psi_{10}(q_0)$. A similar argument shows that $\psi_{10}(r_0) \neq \psi_{10}(p_0)$ and so $\psi_{10}(p_0)$ and $\psi_{10}(q_0)$ are equivalent. Hence the evolution is described by its action on the classical variables which label the equivalence classes and by its restrictions to the corresponding irreducible components.

Let $\tilde{V}_{t_0}$ be the Hilbert spaces associated to the irreducible components of $[O_0, \phi_{01}(I_1)]$ and let V_{t_1} be the corresponding Hilbert spaces associated to the irreducible components of $\mathcal{L}_{t_1}$. If the morphisms are non-trivial, their images are more than a single line, then each component of their restrictions can be represented up to a factor by a semilinear map $U_{10} : \tilde{V}_{t_0} \to V_{t_1}$. Recall that a map $\sigma : V_1 \to V_2$ is called semilinear if it is additive and $\sigma(\alpha f) = s(\alpha)\sigma(f)$, where s is a homomorphism of the underlying fields.

This is a direct application of the following beautiful result proved only recently by Faure and Frölicher [1994]:

Theorem: *Let $g : G_1 \setminus E_1 \to G_2$ be a nontrivial morphism between irreducible projective geometries. Then g induces a semilinear map on the underlying vector spaces V_1 and V_2 with kernel E_1.*

Note that if we ask for some continuity, like just the continuity of the underlying homomorphism $s : \mathbb{C} \to \mathbb{C}$, then s is either the identity or the usual conjugation, or in other words U_{10} is quasilinear (only the axiom of choice can exhibit non-continuous homomorphisms).

Further, if two final states $\psi_{10}(p_0)$ and $\psi_{10}(q_0)$ are orthogonal, then by definition there exists an experimental project α_1 defined at time t_1 which separates them. In principle the experimental project $\Phi_{01}\alpha_1$ then separates p_0 and q_0, since $p_0 < \phi_{01}\psi_{10}(p_0)$ and $q_0 < \phi_{01}\psi_{10}(q_0)$. Thus

Theorem 4.1 *If the motions of p_0 and q_0 are deterministic then*

$$\psi_{10}(p_0) \perp \psi_{10}(q_0) \implies p_0 \perp q_0.$$

In fact this result imposes more stringent conditions on the family of operators U_{10}:

Theorem 4.2 *If the evolution is maximal deterministic and U_{10} is quasilinear then it is unitary (or antiunitary) on a given subspace.*

Proof: The proof rests on the surprising fact that since the map U_{10} preserves orthogonality in the reverse direction, then it must also preserve orthogonality in the forward direction.

First note that U_{10} is injective since it has kernel $\{0\}$ and is in fact bijective on each line due to the fact that as discussed above s is always an isomorphism. Now let f_0 and g_0 be two non-zero orthogonal vectors. We prove that $U_{10}g_0$ and $U_{10}f_0$ are orthogonal. In any case there exists a vector on the line $(U_{10}f_0, U_{10}g_0)$ which is orthogonal to $U_{10}f_0$. Let this vector be called h_1. Since U_{10} is bijective on the line (f_0, g_0) there exists h_0 such that $U_{10}h_0 = h_1$. However, as we have already shown U_{10} always preserves orthogonality in the reverse direction and so h_0 must be orthogonal to f_0. This implies that h_0 is of the form αg_0 and so $U_{10}g_0$ is orthogonal to $U_{10}f_0$ as required.

As is well known, the fact that U_{10} preserves orthogonality in both senses implies that $\langle U_{10}f_0, U_{10}g_0\rangle_1$ is also an inner product on $\tilde{V}_{t_0}$. Since by hypothesis $\tilde{V}_{t_0}$ has dimension at least three this inner product is equivalent to $\langle f_0, g_0\rangle_0$, which means by definition that there exists a real positive number α such that $s^{-1}\big(\langle U_{10}f_0, U_{10}g_0\rangle_1\big) = \langle f_0, g_0\rangle_0\alpha$ for all $f_0, g_0 \in \tilde{V}_{t_0}$, completing the proof. ■

This result justifies the remark that the system either follows a deterministic evolution or consists of a random process: in the case of a maximal deterministic evolution, if the system were to disappear it would do so randomly.

5 Schrödinger Flows

So far we have considered globally the evolution between the initial and final times. Now it is clear that such an evolution must pass through each intermediate time. As one would expect we have the following composition law:

Theorem 5.1 *On* $[O_0, \phi_{02}(I_2)]$ *we have* $\psi_{21}\psi_{10} = \psi_{20}$.

Proof: We start by noting that $\Phi_{01}\Phi_{12} = \Phi_{02}$. This translates the fact that to evolve the system from time t_0 to time t_2 we must pass through the intermediate time t_1 and so the experimental project $\Phi_{02}\alpha_2$ is identical to $\Phi_{01}(\Phi_{12}\alpha_2)$ by definition. This of course implies that $\phi_{01}\phi_{12} = \phi_{02}$ by lemma 3.1. Moreover on $[O_0, \phi_{02}(I_2)]$ we have that $\psi_{21}\psi_{10} = \psi_{20}$ since $p_0 < \phi_{01}\phi_{12}(p_2)$ if and only if $\psi_{10}(p_0) < \phi_{12}(p_2)$. ■

Such a restricted composition law allows us to define a Schrödinger flow in perfect analogy with classical dynamics. Let us write $\Sigma_{t+\tau,t}$ for the domain of $\psi_{t+\tau,t}$ and $\mathcal{U} = \{(\tau, (t, p_t)) \mid p_t \in \Sigma_{t+\tau,t}\}$. Then the Schrödinger flow is the map $W : \mathcal{U} \to \Sigma$ defined by $W_\tau p_t = \psi_{t+\tau,t}p_t$.

Each flow is generated by a field of derivations which is determined by the equation satisfied by the motion. In the classical case these derivations are always exterior and so

given by a vector field [Godbillon 1969, chapter 5]. It is essential in the classical case to make the distinction between a point in the state space Σ and its coordinate functions p, q and t defined on all Σ. In the quantum case, if we represent the state by its one-dimensional projector, any derivation is given by an interior derivation defined by a commutator added to an exterior one defined by a vector field on the classical variables.

The formalism developed above is compatible with many phenomenological equations used in the literature, if we interpret them as equations for certain mean values. This is equally true for some nonlinear equations, where the nonlinearity arises just because we insist to have a normalised vector throughout the process [Piron 1976, §5.3(b)]. Nevertheless, it is possible to make further generalisations, for example to systems where the value of the time is not considered as given *a priori* [Piron 1978].

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