Geometry and quantum symmetries of Calabi-Yau moduli space

Helvetica Physica Acta
Vol. 64 (1991)

0018-0238/91/060811-09$1.50+0.20/0

Geometry and Quantum Symmetries of Calabi–Yau moduli space

S. Ferrara

CERN, Geneva, Switzerland

Some aspects of the compactification of ten-dimensional superstrings and supergravities on Calabi–Yau internal manifolds are discussed.

I. Introduction

I will report on some recent results which have been obtained on the geometrical structure of the moduli space of Calabi–Yau threefolds [1] and their application to compute couplings for effective Lagrangians of supertrings compactified on a $c = 9$ (2,2) superconformad field theory [2].

From the lesson learned from toroidal and orbifold compactifications [3] we know that the moduli space of superconformal field theories is not a smooth manifold but rather a target space orbifold, with fixed points [4]. At these points the effective Lagrangians for the massless modes become singular, in the sense that some massive modes may become massless and therefore they can no longer be integrated out.

It is believed that the moduli space is obtained by modding out a manifold by some discrete (possibly infinite-dimensional) group [5]. The most popular examples are the toroidal compactifications of heterotic superstrings which, in four dimensions, give a moduli space which is $SO(6,22)/SO(6)\times SO(22)/\Gamma$ where Γ is the discrete group $SO(6,22;Z)$ [4].

The smooth manifold $SO(6,22)/SO(6)\times SO(22)$ can be explained from $N = 4$ local space–time supersymmetry arguments while the discrete group Γ, called the duality group, has a (stringy) quantum origin since it is related to the modular invariance of the underlying two-dimensional field theory [4].

The story is much more complicated for Calabi–Yau (C–Y) moduli spaces for which the smooth manifold is not a symmetric space, as for toroidal or orbifold compactifications[1)] and the discrete groups Γ are largely unknown. However, a possible breakthrough has recently been achieved by exploiting the isomorphism between the three-form and even-four cohomology existing at the string level, in the so-called mirror paired C–Y manifolds.

Under these circumstances the quantum symmetries, i.e. the target space dualities of superstring theories, can be studied with topology and algebraic geometry, and a two-dimensional quantum problem can be largely reduced to a classical problem.

---

1) The moduli space of orbifold compactifications is locally a symmetric space for the so-called untwisted moduli [3].

II. Calabi–Yau moduli space and its special 'Hodge–Kähler' geometry

A C–Y threefold is defined as a compact three-dimensional Kähler manifold with vanishing first Chern class; its topological properties are completely characterized by two independent Hodge numbers $h_{(1,1)}$, $h_{(2,1)}$, which also determine the number of 27 and $\overline{27}$ families in heterotic superstring compactifications [6]; $h_{(1,1)}$, $h_{(2,1)}$ are the dimensions of the (Dolbeault) cohomology groups $H^{(1,1)}$, $H^{(2,1)}$. The second and third Betti numbers are respectively given by

$$b_2 = h_{(1,1)} \quad \text{and} \quad b_3 = 2\left(h_{(2,1)} + 1\right) , \tag{1}$$

while $b_0 = b_6 = 1$, $b_4 = b_2, b_1 = b_5 = 0$.

On a C–Y threefold, to the mixed and pure part of the metric fluctuations,

$$\delta g_{i\bar{j}}, \delta g_{ij} \quad (i, j = 1...3) , \tag{2}$$

we can associate elements of $H^{(1,1)}$ and $H^{(2,1)}$. Therefore these fluctuations depend on $h_{(1,1)}$ and $h_{(2,1)}$ parameters, which have the geometrical meaning of moduli parameters.

When a C–Y space is used as an internal threefold for a Kaluza–Klein compactification of a ten-dimensional theory having $M_4 \times C_3$ as a vacuum solution, the moduli parameters appear as massless scalar fields in M_4 (Minkowski space) and the moduli metric, suitably normalized, enters in the scalar field kinetic term

$$G_{AB}(\phi)\, \partial_\mu \phi^A \partial_\mu \phi^B , \tag{3}$$

after integration over the internal manifold degrees of freedom [7]–[9].

The metric G_{AB} can be computed with methods of algebraic geometry since it is related to the scalar product of cohomology representatives (with their dual).

Most surprisingly, G_{AB} does not depend on the explicit knowledge of the metric on C_3 but can be given only in terms of topological informations on the C–Y space.

From independent arguments, based on the deformation theory of complex manifolds, conformal field theory and space–time supersymmetry, one may prove that the moduli space M is (at least locally) a product space [7]–[9]:

$$M = M_1 \times M_2 , \tag{4}$$

where M_1 and M_2 are the moduli spaces for Kähler class and complex structure deformations; moreover M_1 and M_2 must be Hodge–Kähler manifolds of a special type [8], [10].

From the restrictions coming from space–time supersymmetry, the factorization property of M comes from the possibility of using C–Y manifolds as vacua of type II superstrings [8], [10].

The Hodge–Kähler structure means that the Kähler class must be an even element of the integral cohomology, and this comes from the consistent coupling to N = 1 supergravity [11].

The special structure of the two Kähler manifolds M_1, M_2 comes once again from type II superstrings in which the moduli scalar fields belong to N = 2 supermultiplets propagating on M_4.

On a Hodge manifold there is a natural U(1) line bundle with gauge connection [11]

$$Q\,, \quad \text{where} \quad dQ = J \tag{5}$$

is the Kähler form. In components

$$Q = i\left(K_{,i}dz^i - K_{,\bar{i}}d\bar{z}^i\right)\,, \tag{6}$$

where K is the Kähler potential. Under a Kähler transformation

$$K \to K - \Lambda - \overline{\Lambda}\,, \tag{7}$$

the Q connection transforms as

$$Q \to Q + i(\Lambda - \overline{\Lambda}) = Q + \mathrm{Im}\, d\Lambda\,. \tag{8}$$

The U(1) Kähler covariant differential is

$$\nabla\phi = (d + ipQ)\,\phi \tag{9}$$

or in components

$$\nabla_i^{(p)}\phi = \left(\partial_i + \frac{p}{2}K_{,i}\right)\phi,\ \nabla_{i*}^{(p)}\phi = \left(\partial_{i*} + \frac{p}{2}K_{,i*}\right)\phi\,. \tag{10}$$

A covariantly holomorphic field (of weight p) is defined as

$$\nabla_{i*}^{(p)}\widetilde{W} = 0 \tag{11}$$

which implies that

$$\widetilde{W} = e^{\frac{p}{2}K}W\,, \tag{12}$$

where W, with $\partial_{i*}W = 0$, is a holomorphic section of the line bundle.

A special manifold is a Hodge–Kähler manifold whose curvature satisfies the additional constraint [12]–[15]

$$R_{i\bar{j}k\bar{l}} = G_{i\bar{j}}G_{k\bar{l}} + G_{i\bar{l}}G_{k\bar{j}} - C_{ikp}G^{-1p\bar{p}}\overline{C}_{\bar{p}\bar{j}\bar{l}}\,, \tag{13}$$

where C_{ikp} is a covariantly holomorphic symmetric tensor of weight p = 2.

The special geometry implies the existence of h+1 holomorphic sections $L^I(Z)$ (of weight p = 1) such that [14], [15]

$$W_{ikp} = \frac{\partial L^A}{\partial z^i}\frac{\partial L^B}{\partial z^k}\frac{\partial L^C}{\partial z^p}F_{ABC}\,, \tag{14}$$

where $W_{ikp} = e^{-K}C_{ikp}$ and

$$F_{ABC} = \frac{\partial}{\partial L^A}\frac{\partial}{\partial L^B}\frac{\partial}{\partial L^C}F(L)\,; \tag{15}$$

here F is a holomorphic section, homogeneous of second degree in the L^I:

$$L^I\frac{\partial}{\partial L^I}F = 2F\,; \tag{16}$$

the Kähler potential is given by

$$K(z,\bar{z}) = -\log\left(\bar{L}^I(z)\,F_I(z) + L^I(z)\,\bar{F}_I(z)\right), F(z) = F\left(L^I(z)\right) . \tag{17}$$

For deformations of the complex structure the quantities L^A, F_A appear as the 'periods' of the three forms $\Omega(z)$ [8], [9], over the homology cycles $\mathcal{A}^A$, $\mathcal{B}^A$ (A = 1...h+1)

$$L^A(z) = \int_{\mathcal{A}^A} \Omega(z), F_A(z) = -i\int_{\mathcal{B}^A} \Omega(z) \tag{18}$$

or equivalently

$$\Omega(z) = L^A(z)\,\alpha_A - iF_A(z)\,\beta^A , \tag{19}$$

where α_A, β^A is an integral cohomology basis in H^3, dual to the homology cycles. It turns out that

$$W_{ikp} = -i\int_{C_3} \Omega \wedge \frac{\partial^3\Omega}{\partial z^i \partial z^k \partial z^p} \tag{20}$$

satisfies eq. (14) and gives the Yukawa couplings for $\overline{27}$ families in heterotic strings. The Kähler potential [eq. (17)] is

$$K = -\log\; -i\int \Omega\wedge\bar{\Omega} = -\log\; -i\left(\Omega,\bar{\Omega}\right) . \tag{21}$$

The Kähler metric $G_{i\bar{j}}$ is proportional to the scalar product of (2,1) forms ϕ_i

$$G_{i\bar{j}} = -\partial_i\partial_{\bar{j}}\log\; -i\left(\Omega,\bar{\Omega}\right) = -\frac{1}{\left(\Omega,\bar{\Omega}\right)}\left(\varphi_i,\overline{\varphi_{\bar{j}}}\right) , \tag{22}$$

where

$$\varphi_i = \nabla_i\Omega = \partial_i\Omega - \frac{1}{\left(\Omega,\bar{\Omega}\right)}\left(\Omega_i,\bar{\Omega}\right)\Omega ; \tag{23}$$

Equation (17) is also correct for (1,1) moduli, because of N = 2 space–time supersymmetry.

In the large volume limit of C–Y manifolds, the function F is identified with

$$F = i\,d_{ABC}\,L^A L^B L^C / L^0 , \tag{24}$$

where d_{ABC} are the intersection matrices for (1,1) forms. This formula is not valid for quantum C–Y manifolds, owing to non-perturbative (world-sheet) effects [13], [16].

III. Quantum symmetries: target space duality, mirror symmetry and S-map

Recently it has been realized [17] that quantum C–Y manifolds can be constructed in which the role of (1,1) and (2,1) forms is reversed. Such manifolds have been called mirror manifolds.

Since for complex structure deformations the moduli space does not receive (stringy) corrections [13], [16], mirror manifolds allow us to compute the quantum geometry of Kähler class deformations [(1,1) moduli] in terms of the classical geometry of complex structure deformations [(2,1) moduli].

This method also allows us, in principle, to find the quantum duality group Γ alluded to before from algebraic geometrical methods, by studying the periods of the Ω three-form. Recently [18] this method has been applied to the mirror manifold of $P_4(5)$ for which $h_{21} = 1$ and the resulting quantum manifold and duality group Γ have been exactly computed. An illustrative example of mirror symmetry is given by the two-torus [19] which may be regarded as a self-dual mirror manifold with one complex modulus for the Kähler class and one for the complex structure deformation. If one computes its free-energy, i.e. the log of the partition function of a two-dimensional compactification, one obtains [20]

$$F(T,U) = \log\det M_A^2 \,, \tag{25}$$

where the masses of excitations depend on four-integers

$$M_A^2 = \frac{|m + nTU + i(Um' + Tn')|^2}{(T+\overline{T})(U+\overline{U})} \,, \tag{26}$$

and the duality group is SO(2,2;$\mathbb{Z}$), which is a symmetry of the mass spectrum; T, U are complex parameters defined through the three components of the metric and the (single-component) antisymmetric tensor:

$$T = \sqrt{g} + ib, \quad U = \frac{\sqrt{g}}{g_{22}} + i\frac{g_{12}}{g_{22}}, \quad (g = \det g_{ij})\,. \tag{27}$$

The (m,m′) and (n,n′) quantum numbers refer to (quantized) momenta and winding modes.

The classical (Kaluza–Klein) spectrum is obtained by setting $n = n' = 0$. One still notices a residual SL(2,$\mathbb{Z}$) symmetry for complex structure deformation. By using the (quantum) mirror symmetry, i.e. the fact that F(U,T) = F(T,U), one obtains that also the T modulus has an SL(2,$\mathbb{Z}$) symmetry; this has, however, an entirely different origin, since it is based on a symmetry between Kaluza–Klein and winding modes (vortices). The moduli space is in this case

$$\frac{SO(2,2,R)}{SO(2)\times SO(2)} \Big/ SO(2,2;\mathbb{Z}) \tag{28}$$

and the Kähler metric is just the second derivative of the free-energy, i.e. $G_{i\bar{j}} = -\partial_i\partial_{\bar{j}}F$. The free-energy defines an automorphic function of SO(2,2;$\mathbb{Z}$). Actually, performing the sum (after appropriate regularization), one obtains, from eqs. (25) and (26) [20]:

$$F_{\text{regul.}} = +\log|\eta(T)\eta(U)|^4(T+\overline{T})(U+\overline{U})\,, \tag{29}$$

where

$$\eta(T) = e^{-\frac{\pi}{12}T}\prod_n(1-e^{-2n\pi T}) \tag{30}$$

is the 'Dedekind function'.

This example is sufficiently illustrative to suggest the introduction of a generalized 'Dedekind function' for arbitrary Calabi–Yau spaces, which has a deep geometrical meaning [21].

To be more explicit, let us apply the mirror symmetry to Calabi–Yau threefolds; this is an isomorphism between [17]

$$H^3(C,\mathbb{Z}) \quad \text{and} \quad \sum_{i=0}^{3} H^{2i}(C',\mathbb{Z}), \tag{31}$$

of mirror-paired manifolds C, C', i.e. the lattices of integral odd and even cohomology classes.

The action of the duality group Γ on a generic point z of M (z is here either a complex structure or a Kähler class deformation) is such that its action on the 2(h + 1) component vector $(L^A, -iF_A = P_A)$ is an element of $Sp(2(1+h);\mathbb{Z})$. It is then obvious that the holomorphic section $M_A L^A - N^A P_A$, under the action Γz, goes into $M'_A L^A - N'^A P_A$ where the integers (N'^A, M'^A) differ from the original ones by an element of $SP(2(h+1);\mathbb{Z})$.

A globally defined modular invariant quantity is therefore obtained, after regularization, as follows [21]:

$$\sum_{M_A,N_A(\Gamma)} \log \frac{|M_A L^A - N^A P_A|^2}{Y}, \tag{32}$$

where $Y = i\,(L^A \overline{P}_A - \overline{L}_A P^A)$, the sum is meant to be performed on an orbit which therefore depends on Γ.

In this way, the notion of a C–Y superpotential may be introduced, formally defined as the following holomorphic section:

$$W = \prod_{M,N(\Gamma)} (M_A L^A - N^A P_A)\,. \tag{33}$$

As an example, one may apply the above definition to the Z_3/Z_3 orbifold [4], with three-complex moduli, whose geometry is defined by the holomorphic prepotential [8]:

$$F(L^i) = i\frac{L^1 L^2 L^3}{L^0}\,. \tag{34}$$

The final result is [21]

$$\log\frac{|W|^2}{Y}\bigg|_{\text{regul}} = +\log\left(|\eta(T_1)|^4|\eta(T_2)|^4|\eta(T_3)|^4(T_1+\overline{T}_1)(T_2+\overline{T}_2)(T_3+\overline{T}_3)\right), \tag{35}$$

with

$$\begin{aligned} T_i &= -iL_i/L_0 \\ M_0 &= m_1m_2m_3,\ N^0 = n_1n_2n_3,\ M_1 = n_1m_2m_3,\ N^1 = -m_1n_2n_3\,, \\ M_2 &= n_2m_1m_3,\ N^2 = -m_2n_1n_3,\ M_3 = n_3m_1m_2,\ N^3 = -m_3n_1n_2\,. \end{aligned} \tag{36}$$

Equation (35) provides a modular form for the duality group [4]

$$[PSL(2,\mathbb{Z}) \times Z_2]^3 \times S_3 \tag{37}$$

whose interpretation in special geometry is the log of the norm of a holomorphic section of the U(1) bundle, whose first Chern class is the Kähler class of M.

Lastly we would like to point out that the mapping (S-map) [8] of a heterotic theory into a type II theory maps a special Kähler manifold into a 'dual' quaternionic manifold. Dual quaternionic manifolds describe the target-space σ-model geometry of scalar fields in type II theories, unifying moduli fields with Ramond–Ramond scalars.

These manifolds were introduced in ref. [8] and their explicit construction was given in ref. [22], using three-dimensional duality in the original N = 2 supergravity Lagrangian. This has recently been proved to be equivalent to the exact result that one obtains using cohomology theory in type II superstrings [23]. It appears that these quaternonic manifolds were known in the mathematical literature [24] only in the case of homogeneous [25] or symmetric spaces, in which case the covariant derivative of the C tensor vanishes [26]

$$W_{ilmn} = \nabla_i C_{lmn} = 0 . \tag{38}$$

Interestingly enough the Ω tensor [27] of dual quaternonic manifolds is entirely constructed in terms of the following geometrical objects of the associated special Kähler geometry: the Kähler metric, and the C and the W tensors as defined by eqs. (13) and (38).

It would be interesting to understand what the intrinsic definition of dual quaternionic space is, as a condition on the Ω tensor. A derivation of this condition should be possible by analysing the constraints on scattering processes in type II superstrings along the lines of ref. [13].

REFERENCES

[1] E. Calabi, *in* Algebraic geometry and topology (Univ. Press, Princeton, 1957), p. 78; S.T. Yau, Proc. Nat. Acad. Sci. **74** (1977) 1798.

[2] N. Boucher, D. Friedan and A. Kent, Phys. Lett. **B172** (1986) 316;
A. Sen, Nucl. Phys. **B278** (1986) 289 and **B284** (1987) 423;
L. Dixon, D. Friedan, E. Martinec and S.H. Shenker, Nucl. Phys **B282** (1987) 13;
T. Banks and L. Dixon, Nucl. Phys. **B307** (1988) 33;
D. Gepner, Phys. Lett. **B199** (1987) 380 and Nucl. Phys. **B296** (1988) 757.
J. Lauer, D. Lüst and S. Theisen, Nucl. Phys. **B304** (1988) 236 and **B309** (1988) 771.

[3] K. Narain, Phys. Lett. **B169** (1986) 41;
K. Narain, M. Sarmadi and E. Witten, Nucl. Phys. **B279** (1987) 369;
S. Ferrara, C. Kounnas and M. Porrati, Phys. Lett. **B181** (1986) 263;
M. Cvetic, J. Louis and B. Ovrut, Phys. Lett. **B206** (1988) 239 and Phys. Rev. **D10** (1989) 1140;
S. Ferrara and M. Porrati, Phys. Lett. **B216** (1989) 286.

[4] A. Shapere and F. Wilczek, Nucl. Phys. **B320** (1989) 669;
A. Giveon, E. Rabinovici and G. Veneziano, Nucl. Phys. **B322** (1989) 167;
S. Ferrara, D. Lüst, A. Shapere and S. Theisen, Phys. Lett. **B225** (1989) 363;
S. Ferrara, D. Lüst and S. Theisen, Phys. Lett. **B233** (1989) 147;
D. Schevitz, Nucl. Phys. **B338** (1989) 283.

[5] A. Giveon and R. Porrati, Phys. Lett. **B246** (1990) 54 and Nucl. Phys. **B355** (1991) 422.

[6] A. Strominger, Phys. Rev. Lett. **55** (1985) 2547;
A. Strominger, *in* Unifield string theory, eds. M. Green and D. Gross (World Scientific, Singapore, 1986), p. 654;
A. Strominger and E. Witten, Commun. Math. Phys. **101** (1985) 341;
P. Candelas, Nucl. Phys. **B298** (1988) 458.

[7] N. Seiberg, Nucl. Phys. **B303** (1988) 206.

[8] S. Cecotti, S. Ferrara and L. Girardello, Int. J. Mod. Phys. **4** (1989) 24 and Phys. Lett. **B213** (1988) 443.

[9] P. Candelas, P. Green and T. Hübsch, *in* String 88 (eds. J. Gates, L. Preitschof, W. Sergel) (World Scientific, Singapore, 1989);
P. Candelas, T. Hübsch and R. Schimmrigk, Nucl. Phys. **B239** (1990) 582;
P. Candelas and X. De La Ossa, Nucl. Phys. **B355** (1991) 455.

[10] S. Ferrara and A. Strominger, *in* String 89 (Arnowitt, Bryan, Duff, Nanopoulos, Pope, eds.) (World Scientific, Singapore, 1989), p. 245.

[11] E. Witten and J. Bagger, Phys. Lett. **B115** (1982) 202.

[12] E. Cremmer, C. Kounnas, A. Van Proeyen, J.P. Derendinger, S. Ferrara, B. DeWit and L. Girardello, Nucl. Phys. **B250** (1985) 385.

[13] L. Dixon, V. S. Kaplunovsky and J. Louis, Nucl. Phys. **B239** (1990) 27.

[14] A. Strominger, Commun. Math. Phys. **133** (1990) 163.

[15] L. Castellani, R. D'Auria and S. Ferrara, Phys. Lett. **B241** (1990) 57; and Classical Quantum Gravity **7** (1990) 1767;
R. D'Auria, S. Ferrara and P. Fré, preprint CERN TH.5954/90, to appear in Nucl. Phys. B.
[16] M. Dine, P. Huet and N. Seiberg, Nucl. Phys. **B322** (1989) 301.
[17] P. Candelas, M. Lynker and R. Schimmrigk, Nucl. Phys. **B341** (1990) 383.
B.R. Green and R. Plesser, Harvard preprint HUTP–89/B241 and Nucl. Phys. **B338** (1990) 15.
P. Aspinwall, C. Lütken and G. Ross, Phys. Lett. **B241** (1990) 373.
P. Aspinwall and C. Lütken, Nucl. Phys. **B353** (1991) 427 and **B355** (1991) 482.
M. Cvetic, J. Molera and B. Ovrut, Phys. Lett. **B248** (1990) 83;
L. Castellani, M. Monteiro, P. Fré and F. Gliozzi, Turin preprint DFTT/7/90 (1990).
[18] P. Candelas, X. De La Ossa, P. Green and L. Parkes, Phys. Lett. **B258** (1991) 118, and Univ. Texas preprint UTTG–25–1990.
[19] R. Dijkgraaf, E. Verlinde and H. Verlinde, *in* 'Perspectives in string theory', Proc. Niels Bohr Inst. and Nordita Meeting, Copenhagen, 1987, eds. P. Di Vecchia and J. Petersen (World Scientific, Singapore, 1988).
[20] H. Ooguri and C. Vafa, Harvard preprint HUTP–91/A003 (1991).
[21] S. Ferrara, C. Kounnas, D. Lüst and F. Zwirner, preprint CERN TH.6090/91.
[22] S. Ferrara and S. Sabharwal, Nucl. Phys. **B332** (1990) 317.
[23] S. Ferrara, M. Bodner and C. Cadavid, Phys. Lett. **B247** (1990) 25 and Classical Quantum Gravity **8** (1991) 789.
S. Cecotti, Commun. Math. Phys. **131** (1990) 517.
M. Bodner and C. Cadavid, Classical Quantum Gravity **7** (1990) 829.
[24] D.V. Alekseevskii, Math. URSS Izvestija **9** (1975) 297.
[25] S. Cecotti, Commun. Math. Phys. **124** (1989) 23.
B. de Wit and A. Van Proeyen, preprint CERN TH.5865/90.
[26] E. Cremmer and A. Van Proeyen, Classical Quantum Gravity **2** (1985) 445.
[27] J. Bagger and E. Witten, Nucl. Phys. **B222** (1983) 1.