An analytical theory of the anomalous skin-effect in bounded plasmas

Helvetica Physica Acta, Vol. 52 (1979), Birkhäuser Verlag, Basel

An analytical theory of the anomalous skin-effect in bounded plasmas

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(26. IV. 1979)

Abstract. An analytical theory of HF-electromagnetic field distributions in a hot plasma slab or in a hot plasma cylinder, with sharp electron reflecting boundaries is developed. This theory is based on a representation of the solution of the corresponding set of Boltzmann and Maxwell equations by series of functions appropriately chosen for the plane or cylindrical case. Summation of these series by the method of complex integration allows to represent the field vectors by simple closed expressions which reproduce a number of new phenomena observed in experiments with hot plasmas. The existence of a maximum of the energy absorbed in such plasmas, considered as a function of the frequency of the applied field, is established, and limiting regularities describing this effect are formulated.

1. Introduction

Numerical calculations of the electromagnetic fields in a hot plasma slab [1, 2] revealed some new phenomena (e.g. the appearance of a sharp minimum in the spatial distribution of the magnetic field amplitudes) due to the presence of the electron reflecting boundaries. Similar effects were found earlier theoretically also for the anomalous skin effect in a hot plasma half space [6].

Thereupon these effects, connected with the non-local nature of electromagentic phenomena in such plasmas, were investigated experimentally for the bounded cylindrical plasmas [2, 3, 4, 5].

However, a theoretical analysis of the field distributions in cylindrical bounded plasmas is more complicated and, it seems, no theory allowing a direct comparison with the experiments mentioned above was developed so far.

It is an aim of this paper to develop an analytical theory of the electromagnetic field distributions in bounded plasmas accounting both for the non-local relation between current and electric field and for the presence of sharp boundaries. As a basic set of equations we will use a system of Boltzmann equation (in tau-approximation) and Maxwell equations, complemented by a boundary condition characterising the electron interaction with the boundary. Solutions of these equations depending upon the time as $\exp(i\omega t)$ will be represented by infinite series of functions appropriately chosen for the plane or cylindrical case (as it was done already for the plane case in [1, 2]). Summation of these series by methods of complex integration allows then the representation of the field vectors by closed expressions. If both the electron mean free path l and an effective penetration depth δ of the electromagnetic waves are small compared to a plasma dimension a) situation similar to the anomalous skin effect in the plasma half space) these expressions can be simplified essentially and used for comparison with the experiments referred to above. They were shown to be in a satisfactory agreement with experimental results for cylindrical plasmas, described in [4, 5], obtained just under conditions $l \ll a$, $\delta \ll a$. In particular they allow a simple analytical description of the non-monotonic field distributions found experimentally (Sect. 3).

Another interesting conclusion following from our formulas is that the electromagnetic energy absorbed in the hot plasmas (as given by the real part of the surface impedance) can possess, as a function of frequency ω of the applied field, a pronounced maximum (Sect. 4).

2. Plane case

We will consider a plane uniform plasma slab infinite in yz directions and bounded by the planes $x = \pm a$ (Fig. 1). The field in this region is assumed to be generated by two opposing current sheets at $x = \pm a$ so that the magnetic field $H_z = H(x)$ is symmetric about the origin $x = 0$. As a boundary condition we will take $H(x = \pm a) = H_0$. In this case only the field components $E_y = E(x)$, $H_z = H$ differ from zero and, hence, the Boltzmann equation can be written in the form

$$\frac{\partial f}{\partial x} + \gamma f = 2ef_0 v_y E/mu^2 v_x \tag{1}$$

where

$$\gamma = (\nu + i\omega)/v_x, \qquad f_0 = \frac{n_e}{\pi u^2} \exp\left(-\frac{v_x^2 + v_y^2}{u^2}\right), \quad u = \sqrt{2T/m},$$

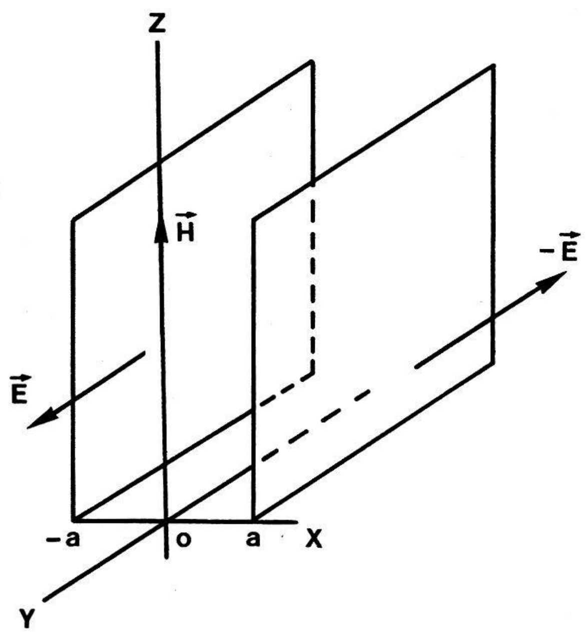

Figure 1
Geometry of the problem for the plane case.

n_e is the free electron density, v_x, v_y are the x and y components of the electron velocity, ν is the electron collision frequency and T is the electron temperature. Accordingly, the Maxwell equations are

$$\frac{\partial E}{\partial x}=ik_0H, \qquad k_0=\omega/c \tag{2}$$

$$\frac{\partial H}{\partial x}=\frac{4\pi e}{c}\int fv_y\,dv_x\,dv_y \tag{3}$$

Introducing the auxiliary functions $\psi^+=\frac{1}{2}(f(v_x,v_y)+f(-v_x,v_y))$, $\psi^-=\frac{1}{2}(f(v_x,v_y)-f(-v_x,v_y))$ (i.e. $f(v_x,v_y)=\psi^++\psi^-$, $f(-v_x,v_y)=\psi^+-\psi^-$) where $v_x>0$, we reduce (1)–(3) to

$$\frac{\partial\psi^+}{\partial x}+\gamma\psi^-=0 \tag{4}$$

$$\frac{\partial\psi^-}{\partial x}+\gamma\psi^+=2ef_0v_yE/v_xu^2m \tag{5}$$

$$\frac{\partial H}{\partial x}=\frac{4\pi e}{c}\int\psi^+v_y\,dv_x\,dv_y, \qquad \frac{\partial E}{\partial x}=ik_0H. \tag{6}$$

The advantage of this representation is evident from the fact, that for specular electron reflection at the boundaries $x=\pm a$, $(f(v_x,v_y)=f(-v_x,v_y)$ for $x=\pm a)$ the boundary condition imposed on the electron distribution function (we will use throughout) can be expressed in a homogeneous form

$$\psi^-(\pm a)=0. \tag{7}$$

The equations (4)–(6) possess the particular solutions $E=A_n\sin(\kappa_nx)$, $H=B_n\cos(\kappa_nx)$, $\psi^-=C_n\cos(\kappa_nx)$, $\psi^+=D_n\sin(\kappa_nx)$, $\kappa_n=\pi n/2a$, satisfying (for odd n) the boundary conditions (7). Inserting these solutions into (4)–(6), we get the relations

$$A_n=\frac{ik_0}{\kappa_n}B_n, \qquad D_n=-\frac{\gamma}{\kappa_n}C_n \tag{8}$$

$$D_n=\frac{2ef_0v_yk_0\gamma i}{\kappa_n(\kappa_n^2+\gamma^2)mu^2v_x}B_n$$

According to (8) the general solutions for ψ^+ and H read as follows

$$\psi^+=\sum_{n=1}^{\infty}D_n\sin(\kappa_nx), \tag{9}$$

$$H=H_0+\sum_{n=1}^{\infty}B_n\cos(\kappa_nx)$$

$$=H_0\sum_{n=1}^{\infty}\frac{2\sin(\kappa_na)\cdot(\cos(\kappa_nx)}{\kappa_na}+\sum_{n=1}^{\infty}B_n\cos(\kappa_nx)$$

and, hence, the equation (6) can be represented in the form

$$\sum_{n=1}^{\infty}\left(\frac{2H_0\sin(\kappa_n a)}{a}+\kappa_n B_n\right)\sin(\kappa_n x)$$
$$=\frac{4\pi e}{c}\sum_{n=1}^{\infty}\int v_y D_n\,dv_x\,dv_y\cdot\sin(\kappa_n x) \qquad (10)$$

leading to the relation

$$2H_0\sin(\kappa_n a)/a+\kappa_n B_n=\frac{4\pi e}{c}\int D_n v_y\,dv_x\,dv_y$$

(We have used here the well known series

$$\frac{4}{\pi}\sum_n\frac{1}{n}\sin\left(\frac{\pi n}{2}\right)\cos\left(\frac{\pi n x}{2a}\right)=1.)$$

It means, that by virtue of (8) the coefficients B_n are given by the formula

$$B_n=H_0\frac{2k_n\sin(k_n\rho)}{\rho D(k_n)},\qquad \rho=\frac{a}{l}, \qquad (11)$$

where

$$D=k_n^2+\frac{\lambda}{k_n}Z\left(\frac{is}{k_n}\right),\qquad k_n=\kappa_n l=\frac{\pi n l}{2a},\qquad l=\frac{u}{\sqrt{\omega^2+\nu^2}},$$

$$Z(\beta)=\pi^{-1/2}\int_{-\infty}^{\infty}\frac{e^{-\xi^2}\,d\xi}{\xi-\beta}$$
$$=-2e^{-\beta^2}\int_{i\infty}^{\beta}dz e^{z^2}=i\sqrt{\pi}e^{-\beta^2}\operatorname{erfc}(-i\beta) \qquad (12)$$

is the plasma dispersion function [7],

$$\lambda=\left(\frac{u}{\nu\delta_0}\right)^2\frac{\omega\nu^2}{(\omega^2+\nu^2)^{3/2}}, \qquad (13)$$

$$\delta_0=\frac{c}{\omega_0},\qquad \omega_0=\sqrt{\frac{4\pi n_e e^2}{m}},\ s=ie^{-i\varepsilon},\qquad \varepsilon=\operatorname{arctg}\frac{\nu}{\omega}.$$

The magnetic field is given, hence, by the series

$$H=H_0S,\qquad S'=\sum_{n=1}^{\infty}\frac{2k_n\sin(k_n\rho)}{\rho D(k_n)}\cos(k_n x),\qquad n\text{ odd}. \qquad (14)$$

(Here the variable x is referred to l, i.e. $-\rho\leqslant x\leqslant\rho$.)

Similarly, the electric field can be represented, by virtue of (2), in the form

$$E=ik_0H_0S,\qquad S=2\sum_{n=1}^{\infty}\frac{\sin(k_n\rho)\sin(k_n x)}{\rho D(k_n)} \qquad (15)$$

For big n (i.e. for big k_n) $D\approx k_n^2$ and hence the series S in (15) converges absolutely. This, in turn, makes it possible to rearrange it in any manner. Thus, we can undertake the following transformations having in mind to bring (15) to the form allowing its summation by the method of complex integration:

$$S=\sum_{n=1}^{\infty}\frac{\sin(k_n\rho)\sin(k_nx)}{\rho D(k_n)}+\sum_{n=-\infty}^{-1}\frac{\sin(k_n\rho)\sin(k_nx)}{\rho D(-k_n)}$$
$$=\sum_{n=-\infty}^{+\infty}\frac{\sin(k_n\rho)\sin(k_nx)}{\rho D(k_n)}+\sum_{n=-\infty}^{-1}\frac{\sin(k_n\rho)\sin(k_nx)}{\rho D(-k_n)}$$
$$-\sum_{n=-\infty}^{-1}\frac{\sin(k_n\rho)\sin(k_nx)}{\rho D(k_n)}=S_1+S_2,$$

where

$$S_1=\sum_{n=-\infty}^{\infty}\frac{\sin(k_n\rho)\sin(k_nx)}{\rho D(k_n)},$$
$$S_2=2i\sqrt{\pi}\lambda\sum_{n=1}^{\infty}\frac{e^{-(is/k_n)^2}\sin(k_na)\sin(k_nx)}{\rho k_nD(k_n)D(-k_n)}$$

(A use was made of the well known formula $Z(\beta)+Z(-\beta)=\sqrt{\pi}ie^{-\beta^2}$ in the last expression.)

The plasma dispersion function Z defined in (11) for positive real k_n can be continued into the whole complex plane ($k_n \to k$) with an exception of the point $k=0$, being an essential singularity (see below). Let us now consider the integral

$$I=I_1+I_2=\frac{1}{2\pi i}\int_{\Gamma}\frac{\sin(kx)\,dk}{D(k)\cos(k\rho)} \tag{17}$$

in the complex plane k, along a path $\Gamma=\Gamma_1+\Gamma_2$ encircling all the roots k_α of the equation $D(k)=0$, the roots of the equation $\cos(k\rho)=0$, and eliminating the singular point $k=0$ as shown in Fig. 2. The integral I_2 along the big circle Γ_2 tends to zero as its radius R_2 tends to infinity because the integrand in I_2 becomes small along this circle (at the least as k^{-2}). The same is true for the integral I_1 along a small circle Γ_1 if its radius R_1 tends to zero. This can be shown by replacing the function $\operatorname{erfc}(-i\beta)$ in (12) by the parabolic cylinder function D_{-1}:

$$\operatorname{erfc}(-i\beta)=\sqrt{\frac{2}{\pi}}\,e^{(1/2)\beta^2}D_{-1}(-\sqrt{2}i\beta),$$

and using for the latter its asymptotic representations valid for $|\beta|=\left|\frac{is}{k}\right|\to\infty$ (see [11], §16.52). We arrive in this way at the following asymptotic formulas:

$$D=k^2-\frac{\lambda}{is},\ \text{for } |\arg(-\sqrt{2}i\beta|<\frac{3\pi}{4}\ \text{and}$$
$$D=k^2-\frac{\lambda}{is}\left(1-2\sqrt{\pi}\frac{1}{k}\exp\left(\frac{s^2}{k^2}\right)\right),$$

elsewhere. (The last expression shows explicitly that the point $k=0$ represents indeed an essential singularity.) We can now use the following inequalities which

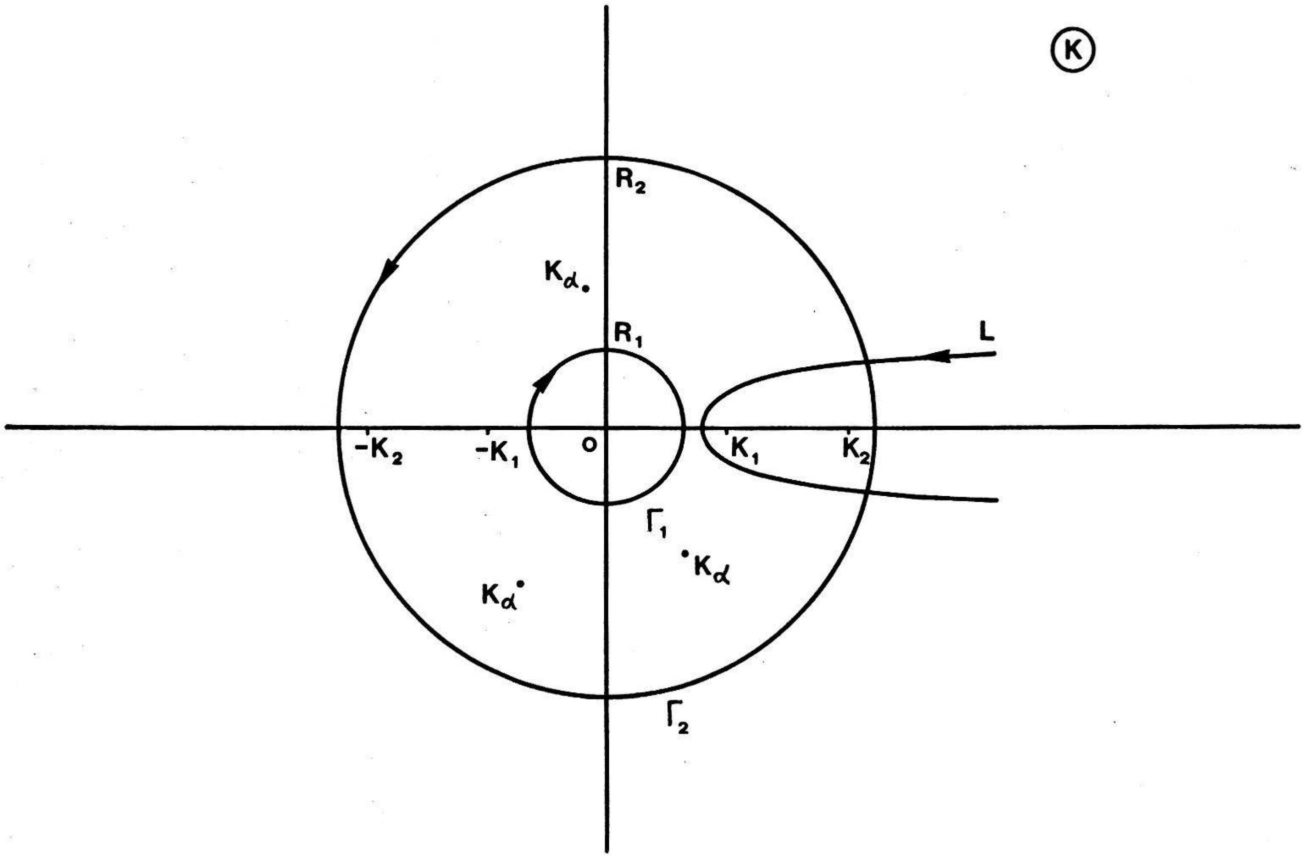

Figure 2
Paths of integration used for summation of the series S_1 and for calculation of the integral J.

demonstrate the convergence of the integral I_1 for $R_1 \to 0$:

$$|I_1| = x\left|\int_{\Gamma_1} \frac{k\,dk}{D}\right| < x\int_{\Gamma_1} \left|\frac{k}{D}\right| |dk| < x\int \left|\frac{k}{k^2 - \dfrac{\lambda}{is}}\right| |dk| \underset{R_1 \to 0}{\to} 0.$$

Thus, we have proved that $I = I_1 + I_2 = 0$ for $R_1 \to 0$, $R_2 \to \infty$. On the other hand, I is equal to the sum of all the residues:

$$I = \sum \operatorname{res} \frac{\sin(kx)}{D(k)\cos(k\rho)} = -S_1 + \sum_\alpha \frac{1}{D'(k_\alpha)} \frac{\sin(k_\alpha \rho)}{\cos(k_\alpha \rho)} = 0, \qquad \text{i.e.}$$

$$S_1 = \sum_{n=-\infty}^{\infty} \frac{\sin(k_n x)}{\rho D(k_n)\sin(k_n \rho)} = \sum_{n=-\infty}^{\infty} \frac{\sin(k_n x)\sin(k_n \rho)}{\rho D(k_n)}$$

$$= \sum_\alpha \frac{1}{D'(k_\alpha)} \frac{\sin(k_\alpha \rho)}{\cos(k_\alpha \rho)}, \qquad k_n \rho = \frac{\pi n}{2}, \qquad (n \text{ odd}).$$

The sum S_1 is, hence, represented in a closed form as a sum of residues corresponding to the roots of the function $D(k)$. (This method of summation is analogous to that described in [8], 4.9.) On the other hand, sum S_2 in (16) is equal to the integral

$$J_1 = \frac{\lambda}{\sqrt{\pi}} \int_L \frac{e^{-(is/k)^2} \sin(kx)\, dk}{kD(k)D(-k)\cos(k_\alpha \rho)} \tag{18}$$

where the path L is chosen as indicated in Fig. 2 in such a way that all zeros of the function $kD(k)D(-k)$ remain outside of this path. (Sum of the residues of the integral in (18) corresponding to the roots of the equation $\cos(k\rho)=0$ coincide, evidently, with S_2.)

Thus, the electric field E is represented by the closed expression

$$E = ik_0 H_0 \left(\sum_\alpha \frac{1}{D'(k_\alpha)} \frac{\sin(k_\alpha x)}{\cos(k_\alpha \rho)} + J_1 \right). \tag{19}$$

As follows from (2), the magnetic field can be also represented by a closed formula

$$H = \frac{1}{ik_0} \frac{\partial E}{\partial x} = H_0 \left(\sum_\alpha A_\alpha \frac{\cos(k_\alpha x)}{\cos(k_\alpha \rho)} + J_2 \right), \qquad J_2 = \frac{dJ_1}{dk}, \tag{20}$$

where $A_\alpha = k_\alpha / D'(k_\alpha)$.

The integrals J_1, J_2 can be evaluated by the saddle point method (see Appendix), while the coefficients A_α have a representation through the roots k_α of the equation $D(k)=0$, which follows easily from (11), (12):

$$A_\alpha = \frac{1}{3}\left(1 + \frac{2}{3}\left(\frac{is\lambda}{k_\alpha^4} + \frac{s^2}{k_\alpha^2}\right)\right)^{-1}. \tag{21}$$

If the parameter λ is big enough, the equation $D(k)=0$ has exactly three roots, which can be represented by an expansion in $\Theta^{-1} = 1/\pi^{1/6}\lambda^{1/3}$:

$$k_\alpha = \Theta\Psi_\alpha - \frac{2s}{3\sqrt{\pi}} + \frac{1}{3}\left(1 - \frac{8}{3\pi}\right)\frac{s^2}{\Theta\Psi_\alpha} + \cdots, \qquad \Psi_1 = i, \quad \Psi_{2,3} = \mp e^{\pm i\pi/6} \tag{22}$$

According to a comparison with the results of numerical calculations of k_α in [6], the series (22) proves to be quite exact (owing to the small factor $\frac{1}{3}(1-8/3\pi) = 0.050$ in the last term) even for not so big parameters $\lambda \gtrapprox 1$.

The complicated integral J_2 in (20) often proves to be small under conditions of interest and the magnetic field distribution is given then by

$$\frac{H}{H_0} = \sum_\alpha A_\alpha \frac{\cos(k_\alpha x)}{\cos(k_\alpha \rho)} \tag{23}$$

or, in the extreme case $\lambda \gg 1$, when $k_\alpha = \Theta\Psi_\alpha$, by

$$\frac{H}{H_0} = \frac{1}{3}\left(\frac{ch\kappa x}{ch\kappa} + 2\,\mathrm{Re}\,\frac{\cos(\kappa e^{i\pi/6}x)}{\cos(\kappa e^{i\pi/6})}\right). \tag{24}$$

(The variable x/l is replaced here by x/a, i.e. in (24) $-1 \leqslant x \leqslant 1$ and $\kappa = \pi^{1/6}\lambda^{1/3}a/l$.) The comparison of (24) with the results of the numerical calculations in [1] shows, that (24) represents satisfactorily, in spite of the drastic simplifications used above, the essential features of the field distribution in the plane case (Fig. 3).

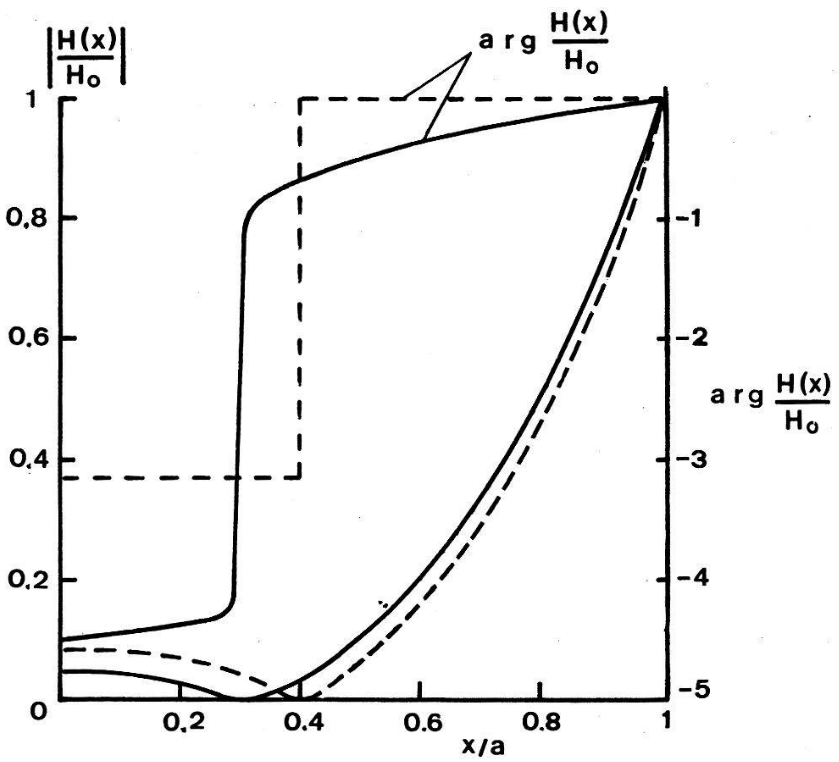

Figure 3
Spatial variation of the amplitude and phase of the magnetic field in the plane case for $\lambda = 3, 7$, $\omega/\nu = 5$, $\varepsilon = 15°$. Full lines represent the results of the numerical calculations in [1], dotted lines were calculated from the formula (24).

3. Cylindrical case

We will consider next the electromagnetic field distribution in an infinite plasma cylinder with a radius $r = a$ under an assumption, that the field is axially symmetric and is generated by a coil at the cylinder surface. The only non-zero components of the fields are now the magnetic component $H_z = H(r)$ parallel to the cylinder axis and the asymutal electric component $E_\Phi = E(r)$. As boundary conditions we will use the requirement $H = H_0$ for $r = a$ and the condition of specular electron reflection (7). The basic kinetic equations must be written in the cylindrical coordinates and read now as follows (see [9]):

$$\nabla_r \psi^+ + \gamma\psi^- + \frac{1}{r} L\psi^- = 0$$

$$\nabla_r \psi^- + \gamma\psi^+ + \frac{1}{r} L\psi^+ = 2ef_0 v_\Phi E / m v_r u^2 \qquad (25)$$

$$L = \left(\frac{v_\Phi^2}{v_r}\frac{\partial}{\partial v_r} - v_\Phi \frac{\partial}{\partial v_\Phi} - 1\right), \qquad \nabla_r = \frac{1}{r}\frac{\partial}{\partial r} r$$

where $\gamma = (\nu + i\omega)/n_r$, v_r, v_Φ are the radial and asymutal components of the electron velocity and f_0, u, $\psi^\pm(v_r, v_\Phi)$ are defined similarily to (1)–(5). The Maxwell equations must be now formulated as:

$$\frac{\partial H}{\partial r}=\frac{4\pi e}{c}\int v_\Phi \psi^+ \, dv_\Phi \, dv_r, \qquad \nabla_r E = ik_0 H \tag{26}$$

Under the conditions of the strong anomalous skin-effect, when the field decays appreciably over a skin depth $\delta \ll a$, and if the mean free path l is small compared to a, one can simplify (25), neglecting the members $L\psi^-$ and $L\psi^-$. (The ratio of these members to the first member in (25) is of the order of $\delta/a \ll 1$ while their ratio to $\gamma\psi^\pm$ is of the order of $l/a \ll 1$. A more detailed justification of this perturbation procedure given in [12], shows that accounting for the members $(1/r)L\psi^\pm$ leads, in fact, to relatively small corrections of the order of δ/a.) The equations (25) are reduced in this way to the form similar to (4), (5):

$$\begin{aligned} &\nabla_r \psi^+ + \gamma\psi^- = 0 \\ &\nabla_r \psi^- + \gamma\psi^+ = 2ef_0 v_\Phi E/mv_r u^2. \end{aligned} \tag{27}$$

The system (26), (27) possess a particular set of solutions satisfying the boundary condition (7): $E = A_n J_1(\kappa_n r)$, $H = B_n J_0(\kappa_n r)$, $\psi^- = c_n J_0(\kappa_n r)$, $\psi^+ = D_n J_1(\kappa_n r)$, where J_0, J_1 are the Bessel function and κ_n are the roots of the equation $J_0(\kappa_n) = 0$. (We employ here the identity $(d/dr) r J_p(\kappa r) = \kappa r J_{p-1}(\kappa r)$.) The substitution of these solutions into (26), (27) allows to find a relation between the coefficients D_n and B_n

$$D_n = \frac{2ef_0\omega v_\Phi \gamma i}{\kappa_n v_r c(\kappa_n^2+\gamma^2)mu^2} B_n. \tag{28}$$

The solutions for H, ψ^+, we look for, are represented by the series:

$$\begin{aligned} &\psi^+ = \sum_{n=1}^{\infty} D_n J_1(\kappa_n r) \\ &H = H_0 + \sum_{n=1}^{\infty} B_n J_0(\kappa_n r) = H_0 \sum_{n=1}^{\infty} \frac{2J_0(\kappa_n r)}{\kappa_n a J_1(\kappa_n a)} + \sum_{n=1}^{\infty} B_n J_0(\kappa_n r) \end{aligned} \tag{29}$$

and, hence, between the coefficients B_n and D_n a relation

$$(2H_0/aJ_1(\kappa_n a) + B_n \kappa_n) = \frac{4\pi e}{c}\int v_\Phi D_n \, dv_\Phi \, dv_r \tag{30}$$

exists, leading by virtue of (28) to an explicit formula for B_n:

$$B_n = \frac{2H_0 k_n}{\rho J_1(k_n\rho) D(k_n)}, \qquad k_n = k_n l, \quad \rho = \frac{a}{l}; \tag{31}$$

(the function $D(k_n)$ is defined by (12)).

The magnetic field is given, accordingly, by the series:

$$\frac{H}{H_0} = \sum_{n=1}^{\infty} \frac{2k_n J_0(k_n r)}{\rho D(k_n) J_1(k_n \rho)}, \tag{32}$$

where the radius r is replaced by r/l.

$\left(\text{We have used the series } \sum_n \frac{2J_0(\kappa_n r)}{\kappa_n a J_1(\kappa_n a)} = 1.\right)$

Repeating the procedure for the plane case verbatim, we reduce (32) to a closed expression:

$$\frac{H}{H_0}=\sum_\alpha A_\alpha \frac{J_0(k_\alpha r)}{J_0(k_\alpha \rho)}+J \tag{33}$$

$$J=\frac{\lambda}{\sqrt{\pi}}\int_L \frac{\exp(-(is/k)^2)J_0(kr)\,dk}{D(-k)D(k)J_0(k\rho)} \tag{34}$$

with coefficients A_α defined by (21), (22). The contribution of the integral J proves to be relatively unimportant under conditions of experiments analysed below (see Appendix). Hence, $H(r)$ is represented under such conditions by the formula:

$$\frac{H}{H_0}=\sum_\alpha A_\alpha \frac{J_0(k_\alpha r)}{J_0(k_\alpha \rho)}. \tag{35}$$

In the extreme case $\lambda \gg 1$, when for the roots k_α and coefficients A_α the asymptotic expressions $k_\alpha=\pi^{1/3}\lambda^{1/3}\psi_\alpha$, $A_\alpha=\frac{1}{3}$ hold, (35) simplifies similarily to

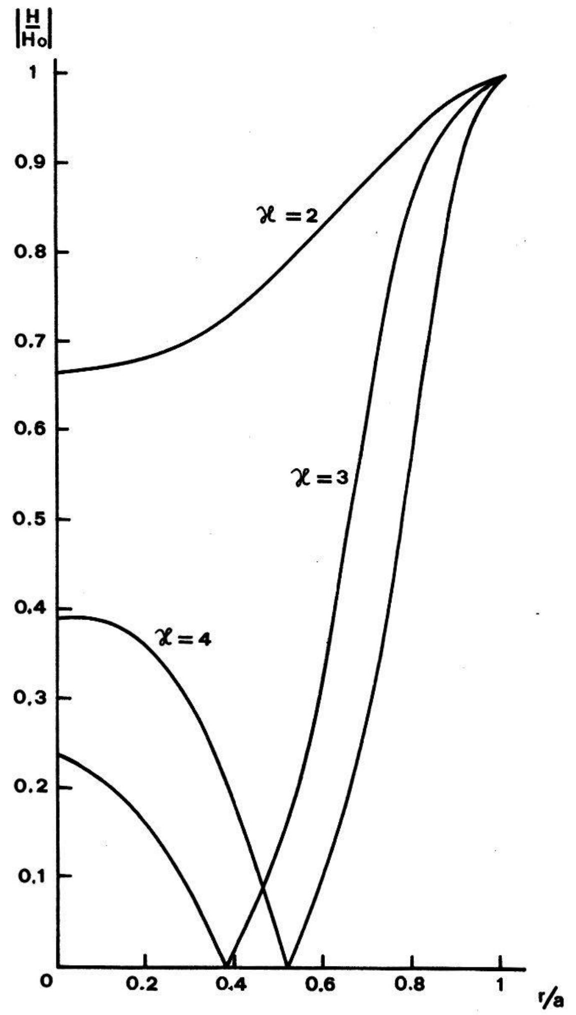

Figure 4
Spatial variation of the amplitude of the magnetic field in the cylindrical case as given by the formula (36) for some values of the parameter $\kappa=(\pi^{1/6}\lambda^{1/3}a/l)$, a is the cylinder radius, $l=u/\sqrt{\omega^2+\nu^2}$.

(24) further:

$$\frac{H}{H_0}=\frac{1}{3}\left(\frac{I_0(\kappa r)}{I_0(\kappa)}+2\,\mathrm{Re}\,\frac{J_0(\kappa e^{i\pi/6}r)}{J_0(\kappa e^{i\pi/6})}\right),\qquad 0\leq r\leq 1. \tag{36}$$

(Parameter κ is defined in (24) and r is referred here to the cylinder radius a.)

The distribution of the magnetic field amplitude defined by (36) is shown for some values of parameter k in Fig. 4. We see from Fig. 4, that in the cylindrical case the same phenomenon arises as in the plane one—the appearance of the minimum of $|H(r)|$ at the point where $H(r)$ (in the framework of the approximation (36)) changes its sign. However, formula (36) can not serve for the quantitative interpretation of the experimental results since the values of $H(r)$ are sensitive to the magnitudes of the roots k_α. The more realistic calculations performed with formula (35) for the experimental conditions used in [4] show (Fig. 5) that the theory developed here conveys satisfactorily all the essential

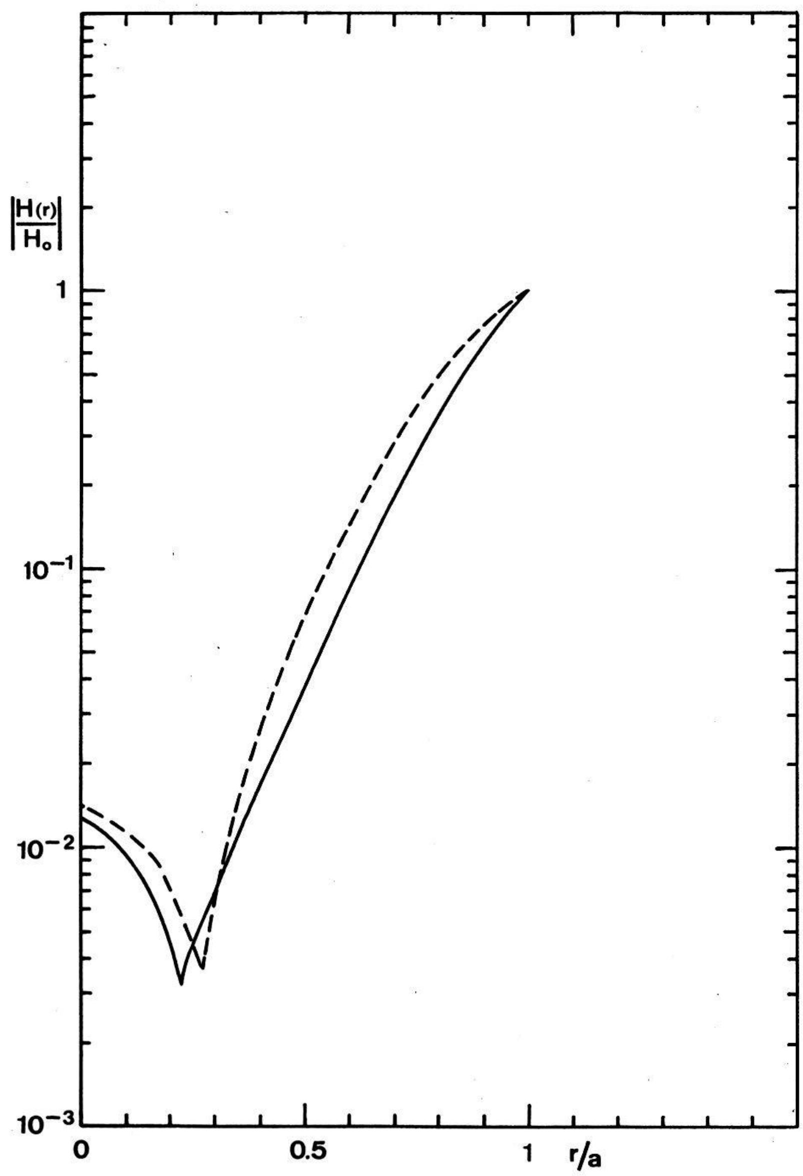

Figure 5

Spatial variation of the magnetic field amplitude in a plasma column under conditions used in ([4], Fig. 5): $a=5$ cm, $\delta_0=0.42$ cm, $T=0$, 6 eV, $\nu=5$, $5\cdot 10^7\ \mathrm{sec}^{-1}$, $\omega/2\pi=7$ MHz. Dotted line is an experimental curve taken from [4] and full line represents the results of calculations with formula (35).

features of the phenomena observed in these experiments, especially the appearance of the sharp minimum in the distribution $|H(r)|$. (Earlier the same conclusion was drawn from a comparison of this theory with the experiments described in [5], see [5, 12]).

An interesting peculiarity of the distributions $|H(r)|$ is the fact that this minimum is not pronounced for low and high values of the frequency ω of the applied field but suddenly becomes sharp at some intermediary values of ω. This fact probably can be explained as follows. The basic parameter λ as defined in (13) and characterising the deviations of the field distribution from the classical one has, as a function of ω, a maximum equal to

$$\lambda_{\max} = \frac{2}{3^{3/2}} \left(\frac{u}{\nu \delta_0} \right)^2 \quad \text{for} \quad \omega_{\max} = \nu/\sqrt{2}.$$

On the other hand, following from formulas (21), (22), (35), the resonance phenomena are the more pronounced the larger the parameter λ is, i.e. these phenomena must appear just at the intermediate values of ω close to $\nu/\sqrt{2}$, where λ is close to the maximal value $\lambda_{\max}$. The inspection of the experimental curves in [4, 5] shows, indeed, that the sharp minimum in $H(r)$ usually appear for $\omega \approx \omega_{\max}$.

The fundamental role of the parameter λ can be clarified as follows. An effective classical penetration depth of the electromagnetic waves is

$$\delta = c|\sqrt{\omega + i\nu}|/\omega\omega_0 \quad (\text{for} \quad \omega \ll \omega_0).$$

The ratio of the effective mean free path $l = u/|\omega + i\nu|$ to this quantity is equal to

$$\frac{l}{\delta} = \frac{u\omega_0}{c} \frac{\omega^{1/2}}{|\omega + i\nu|^{3/2}} = \sqrt{\lambda}.$$

Thus, the parameter λ is a natural measure of non-locality of electromagnetic phenomena in plasmas: these effects are pronounced if the electron mean free path l exceeds $\delta (\lambda \gtrsim 1)$, and they are small otherwise. It is significant that the parameter λ becomes small both for low and high frequencies of the applied field ω—it means that in both these extreme cases the penetration of electromagnetic waves into plasma can be described as a classical skin-effect, while for the intermediate frequencies $\omega \approx \nu$ this effect can be an anomalous one. Of importance is also the appearance of three roots $k_\alpha (\alpha = 1, 2, 3)$ of the equation $D(k) = 0$ for sufficiently big $\lambda \gtrsim 1$; the distribution of magnetic fields in bounded plasmas can be then represented as a superposition of three "fundamental modes": $\cos k_\alpha x$ for the plane case, and $J_0(k_\alpha r)$ for cylindrical case. It is just the interference of these modes which leads to non-monotonic spatial distributions of the magnetic field amplitudes discussed above, this effect being especially strong for big λ, when the roots k_2, k_3 are situated symmetrically relative to the imaginary axis, in the complex plane k.

4. Surface impedance

The formulas for the electromagnetic fields derived above can be used also for a calculation of the average energy W_a (per cm^2 and sec) absorbed in a plasma

$$W_a = \frac{c}{8\pi} \operatorname{Re} E_0 H_0^* = W_0 \operatorname{Re} Z \tag{37}$$

where $Z = E_0/H_0$ is the surface impedance and $W_0 = (c/8\pi)\,|H_0|^2$ is the average energy (per cm^2 and sec) fed into the plasma.

As follows immediately from (19), (33), the surface impedance Z can be represented by the formulas

$$Z = i\frac{\omega l}{c}\left(\sum_\alpha \frac{A_\alpha}{k_\alpha}\tan k_\alpha\rho + \frac{\lambda}{\sqrt{\pi}}\int_L \frac{e^{-(is/k)^2}\tan k\rho}{kD(k)D(-k)}\,dk\right) \tag{38}$$

in the plane case, and by

$$Z = i\frac{\omega l}{c}\left(\sum_\alpha \frac{A_\alpha}{k_\alpha}\frac{J_1(k_\alpha\rho)}{J_0(k_\alpha\rho)} + \frac{\lambda}{\sqrt{\pi}}\int \frac{e^{-(is/k)^2}J_1(k\rho)}{kD(k)D(-k)J_0(k\rho)}\,dk\right) \tag{39}$$

in the cylindrical case, the coefficients A_α being defined in (21). If the parameter $\rho = a/l$ is big enough (thick plasma slab or thick plasma cylinder) we can replace the functions $\tan k\rho$, $J_1(k_\rho)/J_0(k_\rho)$ in (38), (39) by their asymptotic values: $\tan k\rho = \mp i(\mathrm{Im}k \lesseqgtr 0)$, $J_1(k_\rho)/J_0(k_\rho) = \mp i(\mathrm{Im}k \lesseqgtr 0)$. Thus, we get in both cases an expression which coincides with that of surface impedance of a plasma half-space:

$$Z = \frac{\omega l}{c}\left(-\frac{A_1}{k_1} + \frac{A_2}{k_2} + \frac{A_3}{k_3} + \frac{2\lambda}{\sqrt{\pi}}\int_0^\infty \frac{e^{-(is/k)^2}}{kD(k)D(-k)}\,dk\right). \tag{40}$$

(We restrict ourselves here to sufficiently big λ, when the equation $D(k) = 0$ has exactly three roots.)

It is of interest to investigate the frequency dependence of the part of the energy $W_a/W = \mathrm{Re}\,Z$ absorbed in the plasma under the conditions of the anomalous skin effect. A similar analysis was performed earlier for metals in [10] and it was found that Re Z has, as a function of frequency ω, a pronounced maximum. The same conclusion is true also for the hot plasma, in spite of essential differences in the electron distribution functions for metals and gas plasmas. This can be easily demonstrated (as it was done in [12]) by transforming the curves calculated numerically in [6], Fig. 4, and representing the quantity Re Z/ω as a function of λ for different values of the parameter ω/ν. Referring Re Z to Re $Z_0 = \nu/2\omega_0$ and representing Re Z/Re Z_0 as a function of ω/ν (for different parameters $q = u/\nu\delta_0$) we get curves shown in Fig. 6, which demonstrate the presence of a maximum. (Here Re $Z_0 = \nu/2\omega_0$ is the limiting value of Re Z for $\omega/\nu \to \infty$ corresponding to the classical skin-effect. It must be stressed that we use here an approximation $\omega \ll \omega_0$ meaning the neglect of the displacement current in the Maxwell equations. In fact, for very big ω, exceeding ω_0, when the displacement current must be accounted for, Re Z drops to unity in the classical case.)

A question of basic importance which arises naturally in this context is—how big is the maximal part of electromagnetic energy which can be fed into the plasma under the conditions of the anomalous skin-effect?

To answer this question let us simplify the expression (40) using the small parameter $\Theta^{-1} = 1/\pi^{1/6}\lambda^{1/3}$. By developing the factor $1/kD(k)D(-k)$ in the integral in (40) into powers of Θ^{-1} we reduce it to a sum of terms of the type

$$\int_0^\infty \frac{\exp\left[-(is/\Theta x)^2\right]dx}{x^m(x^6+1)^n}$$

(here m, n are some positive powers and $x = k/\Theta$).

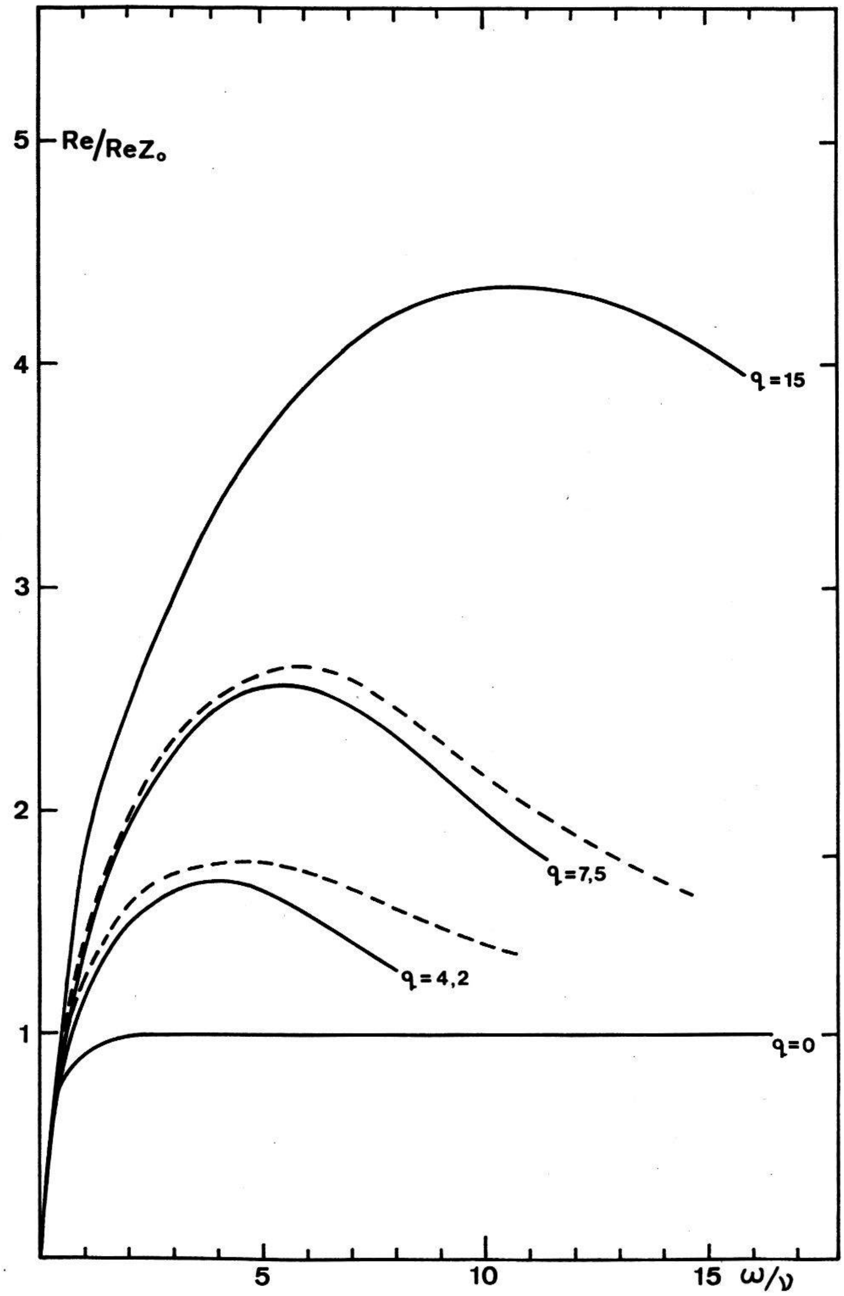

Figure 6
Real part of the surface impedance Z (referred to $\mathrm{Re}\, Z_0 = \nu/2\omega_0$) of a plasma slab or cylinder satisfying the condition $a/l \gg 1$, as a function of ω/ν for different values of the parameter $q = u/\delta_0\nu$. Full lines were calculated with formula (42). For comparison the dotted lines inferred from Fig. 5 in [6] are also reproduced. The curve $q = 0$ corresponds to the normal skin effect.

Owing to the presence of the exponential factor all these integrals are convergent and contain $\ln(\gamma\Theta^{-1})$ for small Θ^{-1} (γ is Euler's parameter). Developing also the ratios A_α/k_α in (40) in powers of Θ^{-1}, we get up to terms of the order of $\Theta^{-3}\ln\Theta$ an expression*

$$Z = \frac{4}{3^{3/2}}\frac{\omega l}{c\Theta}\left[e^{i\pi/3} + \frac{4i}{3\sqrt{\pi}\Theta}\, e^{i(\pi/6-\varepsilon)} + \frac{3^{3/2}}{4\pi}\left(\frac{e^{-i\varepsilon}}{\Theta}\right)^2 \ln\left[\gamma\left(\frac{e^{-i\varepsilon}}{\Theta}\right)^2\right]\right]. \tag{41}$$

* Up to terms of the order of Θ^{-2} (41) coincides with the formula (22) in [6].

Comparison of Re Z as defined by (41) with the curves in Fig. 6 inferred from [6] shows that (41) represents well the results of the numerical calculations if the parameter $q = u/\nu\delta_0$ is big enough. We can hence use (41) to establish the asymptotic regularities corresponding to big q. After some elementary calculations we find in this way the following expression for the maximal value of Re Z for $q \to \infty$:

$$\operatorname{Re} Z = 0{,}123 \frac{u}{c} \quad \text{for} \quad \omega = \omega_{\max} = 0{,}721 \frac{u}{c} \omega_0. \tag{42}$$

In a more general way the regularity (42) can be derived as follows. For big parameters q the maximum of Re Z appears for big ω/ν. One can, hence, simplify λ in (13) neglecting ν compared to $\omega : \lambda = (u\omega_0/c\omega)^2$. Besides, one can take that the parameter $s = ie^{-i\varepsilon}$ entering into (40) is equal to i for $\omega \gg \nu$. Replacing also the factor $\omega e/c$ in (40) by the ratio u/c for $\omega \gg \nu$ one can bring (40) to the form $Z = (u/c)F(\lambda)$ or $\operatorname{Re} Z = (u/c) \operatorname{Re} F(\lambda)$ where $\operatorname{Re} F(\lambda)$ is a function of only one parameter λ, having a maximum $\operatorname{Re} F(\lambda_{\max})$ for some value $\lambda = \lambda_{\max}$. We get in this way again the regularity (42): $(\operatorname{Re} Z)_{\max} \sim u/c$ for $\omega \sim (u/c)\omega_0$. (The numerical coefficients in (42) obtained with the help of the series (41) have an approximate character.)

Thus the maximal part of the electromagnetic energy which can be fed into plasma always remains small, since $u = \sqrt{2T/m} \ll c$. It is connected with the fact that the applied electromagnetic HF-field concentrates in a narrow layer nearby the plasma boundary both for normal and anomalous skin-effect. (However, as follows from (42), for anomalous skin-effect the part of energy absorbed in the plasma is bigger as compared with normal case by the factor $(u/c)/(\nu/\omega_0) = q$.)

In conclusion, it is worth mentioning here that the existence of the maximum of Re Z considered as a function of ω/ν was, at least partly, demonstrated experimentally in [5], Fig. 16.

Acknowledgements

My thanks are due to Dr. B. Joye and Prof. H. Schneider for numerous useful discussions, as well as to Prof. H. Holmann for the valuable mathematical advices. This work was supported by the Swiss National Science Foundation.

Appendix

Integral J in (20) can be calculated (for not too large x) by the saddle point method, in a way similar to that used for the calculation of the "integral along the cut" [6]. For this aim we replace $\cos(k\rho)$ in (20) by $e^{-ik\rho}/2$, and, hence, rewrite it as follows

$$J_2 = \frac{2\lambda}{\sqrt{\pi}} \int_L F(k, x) e^{S(k)}\, dk, \tag{A1}$$

where $F = \cos(kx)/D(k)D(-k)$, $S(k) = -(is/k)^2 + ik\rho$.

The exponent $S(k)$ in (A1) possesses a saddle point $k_s = i(2s^2/\rho)^{1/3}$, which may be considered as belonging to the path L. If ρ is large enough, we can apply for the calculation of the integral J_2 the saddle point method treating $F(k, x)$ as a slowly varying function of k. The standard procedure used in the saddle point method now yields:

$$J_2 = \frac{2i\lambda s^{1/3}}{\sqrt{3}} \frac{\cos(k_s x)}{D(k_s)D(-k_s)} \left(\frac{2}{\rho}\right)^{2/3} e^{-3(\rho s/2)^{2/3}}. \qquad \text{(A2)}$$

For an example considered in Part 2 (Fig. 3) the integral J, as given by (A2), appears small as compared to (20) and thus the use of the very simple formula (24) for calculation of the curves in Fig. 3 is justified. (The value of $|J_2|$ for small $k_s x$ is equal to about 10^{-3} in this case.)

The integral J (34), appearing in the cylindrical case, can be calculated in a similar way by replacing the Bessel function $J_0(k\rho)$ by its asymptotic expression $J_0 = e^{-ik\rho}/\sqrt{2\pi k\rho}$. We reduce in this way the integral (34) to the form

$$J = \frac{\lambda}{\sqrt{\pi}} \int_L F(k, r) e^{S(k)}\, dk, \qquad F = \frac{\sqrt{2\pi k\rho} J_0(kr)}{D(k)D(-k)}. \qquad \text{(A3)}$$

Treating again $F(k, r)$ in (A3) as a slowly varying function of k (this is possible, what concerns $J_0(kr)$, for not too big values of r) we can again calculate J by the saddle point method, leading in this case to the expression

$$J = \frac{i\lambda s^{1/3}\sqrt{2\pi k_s \rho}\, J_0(k_s r)}{\sqrt{3}\, D(k_s)D(-k_s)} \left(\frac{2}{\rho}\right)^{2/3} e^{-3(s\rho/2)^{2/3}} \qquad \text{(A4)}$$

For the case reproduced in Fig. 5 the integral J as given by (A4) appears small compared to the contribution (35) from the residues corresponding to zeros of $D(k)$. Hence, we can use, indeed, the simplified formula (35) for these calculations. In general, the neglect of the integrals J_2, J is possible, as shown by a comparison of (A2) with (23) and of (A4) with (35), if the real part of the exponent in (A2) or (A4), which is equal to $3(\rho/2)^{2/3}\,\mathrm{Re}\, s^{2/3}$, exceeds $\rho\,\mathrm{Re}\, k_\alpha$. This may be the case, if the parameter λ is not too big. Under such conditions (often fulfilled in experiments) the integrals J_2, J contribute for small x/a or small r/a merely a small additive constant, which doesn't change essentially the distributions $H(r)$.

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