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Autor(en): **Jauch, J.M. / Misra, B.**

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The Spectral Representation *)

by **J. M. Jauch** and **B. Misra**

Institute of Theoretical Physics, University of Geneva, Geneva, Switzerland

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Summary. The spectral representation of a countable family of mutually permutable self-adjoint operators is established. It is compared with the representation obtained from the Gelfand isomorphism of an Abelian Banach algebra. The spectral representations as described here can serve as an adequate substitute for the so called «expansion postulate» of quantum mechanics.

1. Introduction

The purpose of this paper is to justify from the mathematical point of view a tool that has often been used in the past by theoretical physicists, viz. the spectral representation of a family of commuting observables. Our analysis will also indicate the limitations of this tool.

The use of the spectral representation (or the closely connected expansion postulate) has been essential in the general formulation of quantum mechanics which replaced and unified the so called *matrix mechanics* of HEISENBERG and the *wave mechanics* of SCHRÖDINGER. The crucial step in this process of unification was the introduction of a transformation theory which allows the rapid transformation from one representation into any other.

DIRAC has developed such a theory in close analogy to the transformation theory in a finite-dimensional linear vectorspace. The Hilbert space of quantum mechanics is however infinite-dimensional and it is not possible to extend this formalism from the finite- to the infinite-dimensional case without a considerable strain on the mathematical notions and a concommittant loss of rigor. DIRAC[1]) needed for this purpose the δ-function together with the system of bra and ket vectors. These elegant formal devices greatly facilitated the development of quantum mechanics and its extension to field theory.

Yet it was clear from the beginning that the δ-function is not a proper function in any sense of the word, but a more general mathematical object which could better be described as a linear functional on a certain class of test functions[2]). There is now an elaborate theory of the δ-function and its generalizations available[3]) and most of the mathematical mysticism of the early transformation theory can, with sufficient effort, be given a meaningful interpretation.

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*) Dedicated to Professor E. C. G. STUECKELBERG von Breidenbach at the occasion of his 60th birthday.

Although with this tool the claim of mathematical rigor can be satisfied, the necessary machinery to implement it, is too cumbersome in most cases, and therefore a better method seems desirable.

Besides, the δ-function is not the only difficulty in DIRAC's formalism. One needs an *expansion postulate* which guarantees that the operators which represent observables admit a sufficiently wide class of eigenfunctions so that every function in Hilbert space can be represented as an unique linear combination of such eigenfunctions. This means that we are dealing with the embedding of a Hilbert space into a Banach space. The correct development of such a theory requires considerable technical tools and has been accomplished only for a very limited class of observables[4])[5]).

The above mentioned difficulties were in principle overcome by VON NEUMANN[6]) who showed in his book that quantum mechanics can be expressed as a mathematical theory avoiding improper functions and the expansion postulate altogether using only the notions of Hilbert space and its linear transformations.

The essential change in the formal apparatus concerns the diagonalization problem of self-adjoint operators. VON NEUMANN shows that the powerful spectral theorem not only has a direct and natural physical interpretation but also that it is sufficiently general to give a completely rigorous mathematical foundation of quantum mechanics.

Yet the problem is solved only in principle. When one works with realistic problems in quantum mechanics one always uses something like an expansion postulate. In fact this seems to be the only *practical* way of evaluating explicitly the spectral projections of the self-adjoint operators encountered in quantum mechanics. Thus it is not surprising that VON NEUMANN's work has hardly influenced the working habits of theoretical physicists.

It was pointed out a long time ago by FRIEDRICHS[7]) that it is neither the spectral theorem nor the expansion postulate which are directly involved in the practical calculations and that one can reconcile the exigences of rigor and usefulness by introducing the notion of the *spectral representation* of self-adjoint operators. This is a particular representation of the Hilbert space as a function space, in which the operators in question appear as multiplication operators. In spite of its usefulness and its obvious content as a generalization of a well-known theorem there does not seem to exist in the literature an explicit proof of sufficient generality of this theorem[8]). For the case of a single operator with simple spectrum the theorem was proved by M. STONE[9]). We are concerned with its generalization to a finite or countably infinite set of commuting self-adjoint operators*).

The method which we have employed is a direct generalization of the method of STONE. For pedagogical reasons we shall not assume this to be known and we shall introduce from the beginning most of the important concepts needed such as spectral measure, functional calcules, cyclic vectors, abelian and maximal abelian von Neumann algebras.

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*) A similar problem was discussed by A. GALINDO in an unpublished thesis (June 1960). The method employed there is restricted to bounded operators. On the other hand GALINDO has obtained results for non-separable Hilbert spaces.

The paper consists of two parts. In part I we discuss in detail the spectral representation for a single operator with simple spectrum. Although the results of this part are not new from a mathematical point of view, we have cast them in such a form that their analogy with the finite-dimensional (trivial) case becomes apparent, and that they can be readily generalized to the new situation of a finite or infinite sequence of operators. This we do in part II. In order not to confuse the main line of the argument, we have suppressed in part II a great deal of detail concerning the extension of the spectral measure on the Cartesian product space. Much of this runs parallel to the corresponding and well-known problem of numerical measures which can be found in the appropriate references.

Throughout both parts we find it convenient to work with the abelian von Neumann algebra generated by the complete set of operators. This raises the question what relation, if any, there may exist between the so-called Gelfand spectral representation of abelian Banach algebras and the spectral representations which we discuss in this paper. This question is examined in the appendix. It will be seen that the beautiful general results of GELFAND are, from the physical point of view, somewhat disappointing. The important applications of the spectral representations always refer to a concrete set of observables and not to the canonical representation of GELFAND. Therefore the spectral representations which we discuss here are perhaps of more direct interest to physicists than the results of GELFAND.

The result, such as it is for instance expressed in theorem 3 of part II has nothing really surprising about it. This result is precisely what every physicist would have expected, who has used the spectral representation as a daily routine. Yet there is one point which merits some comment. We shall see that a family of operators determines a measure class of equivalent measures. If the operators have continuous spectra then this class may or may not contain Lebesgue mesure. In the first case the measure is absolutely continuous with respect to Lebesgue measure. These cases may occur already for one single operator. It would be useful to understand the physical reasons why apart from the discrete case measures which are not absolutely continuous do not seem to occur in physical problems.

Part I

Spectral Representation of Operators with Simple Spectrum

2. An example in finite dimensional space

In order to introduce the essential features of the spectral representation let us discuss first a simple example which is mathematically trivial and therefore known to every physicist.

Let A be a self-adjoint operator in an n-dimensional vectorspace $\mathfrak{H}_n$ over the complex numbers ($n < \infty$). Let φ_r be the eigenvectors of this operator and λ_r the corresponding real eigenvalues. They satisfy equations

$$A \varphi_r = \lambda_r \varphi_r \quad (r = 1, \dots, n) . \tag{1}$$

The λ_r are the n solutions of the secular quation

$$| A - \lambda I | = 0 . \tag{2}$$

If the roots of this equation are all different then the φ_r are pairwise orthogonal and they form therefore a complete system. Since (1) is homogeneous in φ_r we can assume them normalized, so that they form a coordinate system

$$(\varphi_r, \varphi_s) = \delta_{rs} \quad (r, s = 1, \ldots, n) . \tag{3}$$

An arbitrary vector $f \in \mathfrak{H}_n$ can then be written as a linear combination of the vectors

$$f = \sum_{r=1}^{n} x_r \varphi_r , \tag{4}$$

where the complex numbers x_r are determined by

$$x_r = (\varphi_r, f) . \tag{5}$$

Because of Equation (1) we find that

$$A f = \sum_{r=1}^{n} x_r \lambda_r \varphi_r . \tag{6}$$

Thus if we call $\{x_r\}$ the *coordinates* of f in the coordinate system φ_r then we see from the last formula that the coordinates of $A f$ are the numbers $\lambda_r x_r$.

These simple facts are chosen to illustrate the idea of the spectral representation. We wish to reformulate the results just sketched so that they are applicable to self-adjoint operators in Hilbert space ($n = \infty$).

In the general case none of the Equations (1) and (6) can be used since for a continuous spectrum there exist no eigenfunctions in Hilbert space. We must therefore reformulate the essential content of the spectral representation without referring to the eigenfunctions. This can be done if we observe the following point:

The Equation (5) establishes a bijective, linear correspondence between the $f \in \mathfrak{H}_n$ and the sequences $\{x_r\} \in l^2(n)$ (denoted by $f \leftrightarrow \{x_r\}$ in the following).

We define in $l^2(n)$ the addition, multiplication with scalars, and scalar product by the formulae

$$\begin{aligned} \{x_r\} + \{y_r\} &= \{x_r + y_r\} \\ \lambda\{x_r\} &= \{\lambda x_r\} \\ (\{x_r\}, \{y_r\}) &= \sum_{r=1}^{n} x_r^* y_r . \end{aligned} \tag{7}$$

If $f \leftrightarrow \{x_r\}$ and $g \leftrightarrow \{y_r\}$ then

$$f + g \leftrightarrow \{x_r\} + \{y_r\} , \tag{8}$$

$$\lambda f \leftrightarrow \{\lambda x_r\} , \tag{9}$$

$$A f \leftrightarrow \{\lambda_r x_r\} \tag{10}$$

$$\text{and } (f, g) = \sum_{r=1}^{n} x_r^* y_r . \tag{11}$$

Equation (10) characterizes the spectral representation and it shows that in this representation the operator A has an especially simple form. It is a *multiplication operator*.

In this form we have already the germ of a general theorem applicable to a much wider class of operators than those of the starting example. Before we attempt a formulation of a general theorem let us point out some simple and obvious facts which will be useful to keep in mind.

An important question which will come up concerns the *uniqueness* of the spectral representation. It is obvious already in this simple example that the representation is not unique for the simple reason that the Equation (1) determines only subspaces and not individual vectors. Thus if we had chosen for the eigenvectors not the normalized φ_r but another set of vectors, say

$$\psi_r = \frac{1}{\sqrt{\varrho_r}}\, e^{i\alpha r}\, \varphi_r \quad (\alpha \text{ real}, \varrho_r > 0)\,,$$

then we would have obtained another spectral representation, which satisfies also Equations (8), (9), and (10), but instead of (11) we would have obtained

$$(f, g) = \sum_{r=1}^{n} \varrho_r\, x_r^*\, y_r\,. \tag{11}$$

The elements $\{x_r\}$ are thus vectors in a sequence space which we denote by $l_\varrho^2(n)$. Two such spectral representations of the same operator in respective spaces $l_\varrho^2(n)$ and $l_\sigma^2(n)$ are said to be equivalent. There exists then an unitary transformation U from one onto the other defined explicitly by the formulae

$$\{x_r\} \in l_\varrho^2(n) \quad \{y_r\} \in l_\sigma^2(n)$$

$$U\,\{y_r\} = \{x_r\}\,, \tag{12}$$

$$x_r = e^{i\theta r} \sqrt{\frac{\sigma_r}{\varrho_r}}\, y_r\,. \quad (\theta_r \text{ real}) \tag{13}$$

It is thus reasonable to expect that a given operator A defines only a class of *equivalent* spectral representations.

A second remark concerns the notion of the simple spectrum. We have for illustrative purpose assumed that every eigenvalue is non-degenerate. We need evidently a definition of simplicity of the spectrum without referring to the eigenfunction of the operators. If the spectrum is not simple we can only define a spectral representation for a sufficiently large (that is complete) set of commuting operators. In that case it is essential to generalize the theorem to a complete set of commuting operators[10]).

3. *Operators with simple spectrum*

A self-adjoint A in a finite-dimensional space is said to have *simple spectrum* if the system of equations

$$A\,\varphi_r = \lambda_r\,\varphi_r \quad (r = 1, \ldots, n < \infty) \tag{14}$$

has exactly one linear independent solution for every eigenvalue.

This definition is well-known and convenient for the finite-dimensional case but it may become meaningless in the infinite-dimensional Hilbert space. In fact it can be used if and only if the operator A has a pure points spectrum. It is thus indispensable to adopt a definition which has meaning in *all* cases. The procedure for doing this is a standard one: First one introduces a new but equivalent definition for the discrete spectrum which however by a suitable adaptation can directly be transferred to the continuous spectrum.

Let us consider the vector

$$g = \sum_{r=1}^{n} z_r \varphi_r \quad (z_r \neq 0) . \tag{15}$$

It follows from the defining Equation (14) that for any polynomial $p(\lambda)$ we obtain

$$p(A)\, g = \sum_{r=1}^{n} z_r\, p(\lambda_r)\, \varphi_r . \tag{16}$$

Now let x_r be an arbitrary set of complex numbers and set

$$f = \sum_{r=1}^{n} x_r \varphi_r \in \mathfrak{H}_n .$$

If we choose for $p(\lambda)$ a polynomial for which

$$p(\lambda_r) = \frac{x_r}{z_r}$$

then we find

$$p(A)\, g = f . \tag{17}$$

We have thus verified that the vector g has the property: For every $f \in \mathfrak{H}_n$ there exists a polynomial p such that $p(A)\, g = f$. We call a vector g with this property a *cyclic vector*.

It is quite easy to verify that the existence of a cyclic vector is not only a necessary but also a sufficient condition for the spectrum to be simple.

We have thus obtained the desired equivalent definition of a simple spectrum viz. the existence of a cyclic vector. This definition has the great advantage of using no notion which cannot be transferred with only slight modifications to the infinite-dimensional case.

These modifications are essentially of two kinds. First we need a functional calculus which permits the definition of the functions $u(A)$ for a much wider class of functions than the polynomials. Secondly the functions of this class will only generate a linear manifold which is dense in $\mathfrak{H}$.

We shall now introduce the functional calculus in a formal manner.

Definition: A spectral measure on the real line R^1 is a projection valued set function E defined on the Borel subsets B of R^1 with the properties

$$\begin{aligned} &(a) \quad E(\phi) = 0 \qquad (\phi = \text{nul set}) \\ &(b) \quad E(R^1) = I \\ &(c) \quad E\Big(\bigcup_{n=1}^{\infty} \Delta_n\Big) = \sum_{n=1}^{\infty} E(\Delta_n) \qquad \text{whenever } \{\Delta_n\} \end{aligned} \tag{18}$$

is a sequence of mutually disjoint subsets of S.

Every spectral measure has associated with it a functional calculus. This means the following: Every vector $\psi \in \mathfrak{H}$ defines a measure $\mu(\psi)$:

$$\mu_\Delta(\psi) = (\psi, E(\Delta)\, \psi) \quad (\text{for all } \Delta \in \mathfrak{B}) . \tag{19}$$

Let $u(\lambda)$ ($\lambda \in R^1$) be a complex valued function defined almost everywhere (a.e.) and measurable with respect to the measures $\mu(\psi)$. Then it can be shown that there exists one and only one linear operator $u(E)$ with the following properties: The domain of definition D of $u(E)$ consists of those and only those ψ for which

$$\int |\, u(\lambda) \,|^2 \, d\mu(\psi) < \infty . \tag{20}$$

Furthermore, for a $\psi \in D$ and any $\varphi \in \mathfrak{H}$ we define

$$(\varphi, u(E)\, \psi) = \int u(\lambda) \, d\mu\, (\varphi, \psi) , \tag{21}$$

where $\mu(\varphi, \psi)$ denotes the complex valued measure defined by

$$\mu_\Delta\, (\varphi, \psi) = (\varphi, E(\Delta)\, \psi) . \tag{22}$$

The formula (21) defines an unique vector $\psi' = u(E)\, \psi$ and the transformation $\psi \to \psi'$ is easily seen to be linear.

By means of Equations (19), (20), and (21) we have thus established a correspondence between certain complex valued functions $u(\lambda)$ defined on the real line and operators in Hilbert space. This correspondence is called the *functional calculus* of the spectral measure E.

Every self-adjoint operator A defines an unique spectral measure and vice versa, such that the diagonal elements of the operator A permit the representation

$$(f, A\, f) = \int_{-\infty}^{+\infty} \lambda \; d\, (f, E\, (\lambda)\, f) \tag{23}$$

for every f in the domain of A, where $E(\lambda) \equiv E\,\{(-\infty, \lambda)\}$ is the projection associated with the interval $(-\infty, \lambda)$.

This is the spectral theorem. Because of this unique correspondence we shall also write $u(A)$ instead of $u(E)$ for the operator defined by Equation (21).

Let us now consider a fixed vector $g \in \mathfrak{H}$. The set of all the functions $u(A)$ for which $u(A)\, g$ exists, defines a linear manifold $\{u(A)\, g\}$. We denote the closure of this manifold by

$$M(g) = \overline{\{u(A)\, g\}} \tag{24}$$

and we say that the vector g is *cyclic* (with respect to A) if $M(g) = \mathfrak{H}$.

All this is needed for the formulation of the following

Definition: A self-adjoint operator A is said to have *simple spectrum* if there exists a cyclic vector for A.

Speaking now informally we may say that the operators with simple spectrum are characterized as operators with maximal mobility. The identity operator has an infinitely degenerate eigenvalue 1. It leaves every vector invariant. The operators with degenerate spectrum are identity operators with respect to the subspaces belonging to the same eigenvalues. Thus a vector in such a subspace can never be moved around by any function of A. If there are no such subspaces then the operator A has maximal mobility and there exists a cyclic vector. This is the heuristic meaning of the definition.

For the rest of this section we shall describe another equivalent definition of an operator with simple spectrum. In this definition we shall avoid any explicit reference to the functional calculus and replace it by purely algebraic considerations. As an unexpected bonus one obtains a dual aspect of operators which permits to establish several of the important theorems as pairs of dual theorems.

The central notion in this approach is the *commutant*. If A is a bounded operator, we say another bounded operator B commutes with it if $AB = BA$. The extension of this definition to the unbounded case needs some care: A bounded operator B commutes with an unbounded operator A if $BA \subset AB$. This means the operator AB is an extension of the operator BA.

Consider now a family $\mathfrak{S}$ of bounded or unbounded operators. We define the commutant $\mathfrak{S}'$ as the set of all *bounded* operators which commute with all the operators in $\mathfrak{S}$.

The commutant $\mathfrak{S}'$ is an algebra and it is weakly closed. Furthermore if $\mathfrak{S}$ is invariant under the adjoint operation (for instance if $\mathfrak{S}$ contains only self-adjoint operators) then $\mathfrak{S}'$ contains with every operator B its adjoint. $\mathfrak{S}'$ is thus a von Neumann algebra.

It is clear that $(\mathfrak{S}')' \equiv \mathfrak{S}'' = \mathfrak{N}$ is also a von Neumann algebra and we shall call it the algebra *generated* by $\mathfrak{S}$. It is the smallest von Neumann algebra containing all the spectral projections of the operators in $\mathfrak{S}$. If $\mathfrak{N} \subset \mathfrak{N}'$ then $\mathfrak{N}$ is said to be abelian. $\mathfrak{N}$ is abelian if and only if every element of $\mathfrak{S}$ commutes with every other one[12]).

If $\mathfrak{S}$ consists of a single self-adjoint operator A then the abelian algebra $\mathfrak{A} = \{A\}''$ generated by A consists of all bounded functions of A[11]). Since it is abelian we have always

$$\mathfrak{A} \subseteq \mathfrak{A}' \tag{15}$$

If $\mathfrak{A} = \mathfrak{A}'$ we say $\mathfrak{A}$ is *maximal* abelian. It admits then no abelian extensions.

It is clear that if $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are two von Neumann algebras such that $\mathfrak{N}_1 \subseteq \mathfrak{N}_2$ then we have $\mathfrak{N}_2' \subseteq \mathfrak{N}_1'$. Thus qualitatively speaking: The larger the algebra, the smaller is its commutant.

It comes therefore as no surprise that we have the following

Proposition: An abelian von Neumann algebra admits a cyclic vector if and only if it is maximal ablian[12]).

This theorem establishes the bridge between the old and the following equivalent definition of the simple spectrum

Definition: A self-adjoint operator is said to have *simple spectrum* if it generates a maximal abelian von Neumann algebra.

4. *The spectral density function*

An important step in the process of establishing a spectral representation is the construction of a certain measure, or more precisely a measure class, defined on the Borel sets of the real line.

In the following we shall construct this measure for a self-adjoint operator A with simple spectrum. All the measures which we consider will be defined on the Borel sets of the real line. The spectrum of A will be denoted by Λ. It may be discrete, continuous, or mixed. We denote by $\Delta \rightarrow E(\Delta)$ the spectral measure defined by the operator A. In particular for the spectral family we write $E_\lambda \equiv E((-\infty, \lambda])$ for all real λ. It will occasionally be useful to distinguish classes of equivalent sets with respect to the spectral measure. Two sets Δ_1 and Δ_2 are equivalent if $E(\Delta_1) = E(\Delta_2)$. It is easily seen that this is an equivalence relation which therefore defines a class structure on the σ-ring of all Borel sets. We shall denote by $\{\Delta\}$ the class of all equivalent sets which contains a particular set Δ. The equivalence classes are partially ordered as follows: We say $\{\Delta_1\} \subset \{\Delta_2\}$ if and only if $E(\Delta_1) \leqslant E(\Delta_2)$. We say two classes $\{\Delta_1\}$ and $\{\Delta_2\}$ are *orthogonal* if $E(\Delta_1)\, E(\Delta_2) = 0$. For any given unit vector φ we can define two objects; a measure on the Borel sets of the real line and a subspace of $\mathfrak{H}$. They are defined as follows:

First we introduce the spectral density function $\sigma(\lambda)$ by the formula

$$\sigma(\lambda) = (\varphi, E_\lambda\, \varphi)\ . \tag{25}$$

This function has the following properties

$$\begin{aligned} &\text{(a)} \quad \sigma(\lambda_1) \leqslant \sigma(\lambda_2)\ , \\ &\text{(b)} \quad \sigma(\lambda + 0) = \sigma(\lambda)\ , \\ &\text{(c)} \quad \sigma(-\infty) = 0\ , \\ &\text{(d)} \quad \sigma(+\infty) = 1\ . \end{aligned} \tag{26}$$

It is thus a non-decreasing, right continuous (but not necessarily continuous) function, normalized to 0 and 1 at $-\infty$ and $+\infty$ respectively.

The function $\sigma(\lambda)$ defines a measure, by assigning to every Borel set Δ the measure

$$\sigma(\Delta) = \int_\Delta d\sigma(\lambda) = (\varphi, E(\Delta)\, \varphi) \tag{27}$$

The subspace $M(\varphi)$ associated with φ is defined by

$$M(\varphi) = \overline{\{\mathfrak{A}\, \varphi\}}\ , \tag{28}$$

where $\mathfrak{A}$ denotes the von Neumann algebra generated by the self-adjoint operator A, and the right-hand side denotes the closure of the linear manifold of all elements of the form $T\,\varphi$ with $T \in \mathfrak{A}$.

There is an intimate relation between the measure σ and the subspace $M(\varphi)$. Both have an identical partial order structure as follows: We say a measure σ_1 is *inferior* to another measure σ_2 and we write $\sigma_1 \propto \sigma_2$ if every set of σ_2-measure zero is also of σ_1-measure zero.

We say that σ_1 is *equivalent* to σ_2 if both $\sigma_1 \propto \sigma_2$ and $\sigma_2 \propto \sigma_1$. This is an equivalence relation and defines thus the classes of equivalent measures. We denote by $\{\sigma\}$ the class of equivalent measures which contains a particular measure σ.

The measures σ_1 and σ_2 are said to be *orthogonal*, and we write $\sigma_1 \perp \sigma_2$ if there exist two disjoint measurable sets A and B such that

$$A \cup B = \{R^1\} \text{ and such that } \sigma_1(A) = \sigma_2(B) = 0\,. \tag{29}$$

The following theorem asserts that the two ordering relations are identical.

Theorem: Let A be a self-adjoint operator with simple spectrum, E_λ its spectral family, φ_1 and φ_2 two unit vectors in $\mathfrak{H}$, σ_1 and σ_2 the measures associated with the spectral density functions

$$\sigma_1(\lambda) = (\varphi_1, E_\lambda\, \varphi_1),\ \sigma_2(\lambda) = (\varphi_2, E_\lambda\, \varphi_2)$$

and

$$M_1 = M(\varphi_1),\ M_2 = M(\varphi_2)\,.$$

Then $\sigma_1 \propto \sigma_2$ if and only if $M_1 \leqslant M_2$ and $\sigma_1 \perp \sigma_2$ if and only if $M_1 \perp M_2$.

We shall need.

Lemma 1: The projection P with range $M(\varphi)$ is contained in $\mathfrak{A}'$.

Lemma 2: Every projection $P \in \mathfrak{A}$ determines uniquely a class of equivalent sets $\{\varDelta\}$ and conversely. The correspondence $P \leftrightarrow \{\varDelta\}$ preserves the ordering and orthogonal classes correspond to orthogonal projections.

Lemma 1 is proved on page 5 of reference [11]). We proceed to prove lemma 2. According to a known theorem [13]) every operator P in $\mathfrak{A}$ is a function of A, so that

$$P = \int_{-\infty}^{+\infty} u(\lambda)\; dE_\lambda \tag{30}$$

with some Borel function $u(\lambda)$.

Since P is a projection, we must have $u^2(\lambda) = u(\lambda)$ a.e., or

$$u(\lambda) = \begin{cases} 1 \\ 0 \end{cases}.$$

Let $\varDelta$ be the set with the property

$$u(\lambda) = \begin{cases} 1 & \text{for } \lambda \in \varDelta \\ 0 & \text{for } \lambda \notin \varDelta \end{cases}$$

It follows that $\varDelta$ is a Borel set and $P = E(\varDelta)$. The equivalence class $\{\varDelta\}$ is uniquely determined by P, furthermore if $P_1 < P_2$ then $\{\varDelta_1\} < \{\varDelta_2\}$ and if $P_1 P_2 = 0$ then $\{\varDelta_1\} \perp \{\varDelta_2\}$. This proves lemma 2.

Proof of the theorem

Assume $\sigma_1 \propto \sigma_2$. We wish to show $M_1 \leqslant M_2$. Let $\varphi_1 = u + v$ with $u \in M_2$ and $v \in M_2^\perp$. Define $N = \{\mathfrak{A}\, v\}$ and denote by Q the projection with range N. According to lemma 1 $Q \in \mathfrak{A}'$ and since $\mathfrak{A} = \mathfrak{A}'$, $Q \in \mathfrak{A}$. According to lemma 2 there exists an unique class of equivalent sets $\{\varDelta\}$ such that $Q = E(\varDelta)$. Since $QM_2 = 0$ we have $\sigma_2(\varDelta) \equiv (\varphi_2, E(\varDelta)\, \varphi_2) = 0$. By assumption it follows that $\sigma_1(\varDelta) \equiv (\varphi_1, E(\varDelta)\, \varphi_1) = 0$. Consequently $E(\varDelta)\, \varphi_1 = Q\, \varphi_1 = Qv = v = 0$. Thus $\varphi_1 \in M_2$. If P_2 is the projection onto M_2 we have $T\varphi_1 = TP_2\varphi_1 = P_2 T\varphi_1 \in M_2$ for all $T \in \mathfrak{A}$.

It follows that $\{\mathfrak{A}\varphi_1\} = M_1 \leqslant M_2$. q.e.d.

Assume conversely that $M_1 \leqslant M_2$. Let Δ be a Borel set such that $\sigma_1(\Delta) \neq 0$, that is $E(\Delta)\,\varphi_1 \neq 0$. We wish to prove that $\sigma_2(\Delta) = (\varphi_2, E(\Delta)\,\varphi_2) \neq 0$. Suppose to the contrary that $\sigma_2(\Delta) = 0$, or equivalently $E(\Delta)\,\varphi_2 = 0$. Since $E(\Delta) \in \mathfrak{A} = \mathfrak{A}'$ we also have $E(\Delta)\,T\varphi_2 = 0$ for all $T \in A$. Thus $E(\Delta)\,M_2 = 0$. This contradicts $E(\Delta)\,\varphi_1 \neq 0$. Thus $\sigma_2(\Delta) \neq 0$. This means $\sigma_2(\Delta) = 0$ implies $\sigma_1(\Delta) = 0$, or $\sigma_1 \propto \sigma_2$. q.e.d.

This proves the first half of the theorem. The second half can be proved with similar techniques. Since we do not need the second half in the following we shall omit its proof here[13]).

From the first half of the theorem alone follows that there exists an unique measure class $\{\varrho\}$ generated by a vector φ with the property that any other such measure σ is inferior to $\varrho : \sigma \propto \varrho$.

For the measure ϱ the corresponding vector φ is cyclic, so that $M(\varphi) = \mathfrak{H}$.

Concerning the nature of the measure ϱ we add a few statements without proof.

Let Δ_i be the family of open sets such that $\varrho(\Delta_i) = 0$. The union $\Delta_0 \equiv \underset{i}{U}\,\Delta_i$ is then also open and its complement $\Lambda = \Delta_0'$ is closed. It is called the spectrum of A. The discrete part of the spectrum consists of the points of discontinuities of $\varrho(\lambda)$, that is those points λ_i for which $\varrho(\lambda_i - 0) < \varrho(\lambda_i)$. The remaining part of the spectrum can be further decomposed into the absolutely continuous part and the singular part.

We may write

$$\varrho = \varrho_d + \varrho_a + \varrho_s \,.$$

To this decomposition corresponds a decomposition of the Hilbert space into the direct sum of three orthogonal subspaces

$$\mathfrak{H} = \mathfrak{H}_d + \mathfrak{H}_a + \mathfrak{H}_s \,.$$

The first two parts occur often in physical problems (for instance in the hydrogen atom problem). The singular part has so far never occurred in any physical problem.

5. *The spectral representation*

We have now all the tools on hand to establish the spectral representation for a self-adjoint operator with simple spectrum. Let us recall what is our objective. We desire to find a generalization of Equation (10). That is we wish to construct a function space L_p^2 over the reals and establish a norm preserving correspondence (isomorphism) between the abstract space $\mathfrak{H}$ and $L_p^2(\lambda)$ such that the operator A is represented in L_p^2 as a multiplication operator.

Our task is accomplished by carrying out the following steps:

(1) Establish the measure class $\{\varrho\}$.

(2) Establish the correspondence $f \leftrightarrow \{u(\lambda)\}$.

(3) Verify linearity and isometry of this correspondence.

(4) Verify the spectral representation of the operator A.

We are thus given a self-adjoint operator A with simple spectrum operating in a dense domain of definition $D_A \leqslant \mathfrak{H}$. The operator A may be bounded or unbounded. Since it has simple spectrum there exist cyclic vectors g and by the theorem of the preceding section these vectors determine an unique measure class $\{\varrho\}$ on the spectrum of the operator. This is the measure class to be established according to point (1) of the above program.

Let us proceed to point (2). Let $\mathfrak{M} = \{\mathfrak{A}\, g\}$ be the dense linear manifold generated by a cyclic vector g and let $f \in \mathfrak{M}$. It is thus of the form $f = T\, g$ with $T \in \mathfrak{A}$. Every $T \in \mathfrak{A}$ can be represented in the form [13])

$$T = \int_{-\infty}^{+\infty} u(\lambda)\, dE_\lambda \tag{31}$$

and the norm of T is given by the formula

$$\| T \| = \text{ess. sup}\, \{u(\lambda)\}\,. \tag{32}$$

Since the norm of every $T \in \mathfrak{A}$ is finite, every element f determines a.e. an unique essentially bounded function $u(\lambda)$. Conversely every such function determines a T through the formula (31). If we denote by $\mathfrak{L}$ the set of all functions over Λ we have constructed a one-to-one correspondence between the elements of $\mathfrak{M}$ and those of $\mathfrak{L}$ which preserves linearity. This can be extended by continuity to the closure of $\mathfrak{M}$ and of $\mathfrak{L}$; Since $\mathfrak{L}$ is dense in L^2_p and $\mathfrak{M}$ is dense in $\mathfrak{H}$ we have established the desired mapping of $\mathfrak{H}$ onto L^2_p.

Concerning point (3) we observe that the linearity is an immediate consequence of the definition. The isometry is contained in the formula

$$\| f \|^2 = \int_{-\infty}^{+\infty} | u(\lambda) |^2\, d\varrho(\lambda)\,. \tag{33}$$

Let us finally verify (4). To this end we observe that the correspondence $u(\lambda) \to u(A)$ of the functional calculus is *multiplicative*. This means if

$$u(\lambda) = u_1(\lambda)\, u_2(\lambda)\,,$$

then

$$u(A)\, f = u_1(A)\, u_2(A)\, f$$

for every f in the domain of

$$u_1(A)\, u_2(A)\, {}^{*})$$

Let now f be in the domain of A and let

$$f = u(A)\, g\,.$$

Thus g is also in the domain of $A\, u(A)$ and therefore

$$A\, f = A\, u(A)\, g = W(A)\, g\,,$$

where $w(\lambda) = \lambda\, u(\lambda)$. This result shows that if $u(\lambda)$ represents the vector $f \in D_A$ then $\lambda\, u(\lambda)$ represents the vector $A\, f$. We can express this relation in the form

$$A\, \{u(\lambda)\} = \{\lambda\, u(\lambda)\}\,. \tag{34}$$

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*) This remains true even for unbounded operators cf. reference [17]), p. 346.

With this result we have established the spectral representation of operators with simple spectrum.

Part II

Spectral Representation of a Family of Commuting Operators

In this part we shall establish the spectral representation of a complete set of commuting observables which may be a finite or a countably infinite set. The main tool will be a certain spectral measure defined on the product space of the spectra of the operators. The subsequent arguments leading to the spectral representation are nearly the same as in part I. The only departure occurs while implementing the point 4 of the program outlined in the last section.

6. *The spectral measure of mutually commuting operators*

Let $\mathfrak{S} = \{A_i\}$ $(i = 1, 2, \ldots)$ be a complete set of mutually commuting self-adjoint operators. Each operator A_i has associated with it a spectrum Λ_i and a spectral measure P_i. Our method of constructing the spectral measure of $\mathfrak{S}$ from those of the operators A_i is modelled after that of constructing the product measure from a set of ordinary (i.e. numerical valued) measures[14]).

In order to formulate the principal theorem of this section properly we need a number of definitions.

If $\{\Lambda_i\}$ is a (finite or countably infinite) sequence of sets then the set of all ordered sequences $\{\lambda_1, \lambda_2, \ldots, \lambda_i, \ldots\}$ where $\lambda_i \in \Lambda_i$ is called the *product set* of $\{\Lambda_i\}$. It will be denoted by $\underset{i}{\mathrm{X}}\, \Lambda_i$.

A subset of $\underset{i}{\mathrm{X}}\, \Lambda_i$ which is of the form $\underset{i}{\mathrm{X}}\, M_i$, where $M_i \leqslant \Lambda_i$ and $M_i = \Lambda_i$ for all but a finite number of values of i will be called a *rectangle* of $\mathrm{X}\, \Lambda_i$. The sets M_i will be called the *sides* of the rectangle.

If Λ_i is a topological space with a family τ_i of open subsets then a topology can be introduced in $\underset{i}{\mathrm{X}}\, \Lambda_i$ in the following manner: Consider the collection β of all sets of the form $\underset{i}{\mathrm{X}}\, U_i$ where $U_i \in \tau_i$ and $U_i = \Lambda_i$ for all but a finite set of indices i.

Then the family τ of all sets which may be obtained as the union of some subfamily of β can be taken as the family of open subsets of $\underset{i}{\mathrm{X}}\, \Lambda_i$. The topology defined by this class of open subsets is called the *product topology*.

If Λ_i is a locally compact Hausdorff space then the sets belonging to the σ-ring generated by the compact subsets of Λ_i will be called the *Borel sets of* Λ_i.

In the case of the real line with the usual topology, the class of all its Borel sets coincides with the σ-ring generated by all open sets.

A rectangle $\underset{i}{\mathrm{X}}\, M_i$ of $\underset{i}{\mathrm{X}}\, \Lambda_i$ will be called a *Borel rectangle* if every side M_i is a Borel set of Λ_i.

A set function P defined on a class τ of sets is said to be *additive* on τ if for every pair M and N of disjoint sets in τ the union $M \cup N$ of which is also in τ, we have

$$P\,(M \cup N) = P(M) + P(N)\,.$$

We shall say P is *finitely additive* on τ if for every finite class of mutually disjoint sets M_i $(i = 1, \ldots, n)$ with the property that $\bigcup_i M_i$ is also in τ

$$P\left(\bigcup_{i=1}^{n} M_i\right) = \sum_{i=1}^{n} P(M_i) .$$

Finally the set function P will be said to be *σ-additive* on τ if for every infinite sequence M_i of mutually disjoint sets such that $\bigcup_{i=1}^{\infty} M_i \in \tau$ we have

$$P\left(\bigcup_{i=1}^{\infty} M_i\right) = \sum_{i=1}^{\infty} P(M_i) .$$

The construction of the spectral measure on the product space proceedes now as follows: With every Borel rectangle $\underset{i}{\mathrm{X}} M_i$ in the product space $\underset{i}{\mathrm{X}} \Lambda_i$ we can associate a projection operator

$$P' \,(\underset{i}{X} M_i) \equiv \operatorname{slim}_{n \to \infty} \prod_{i=1}^{n} P_i(M_i) \equiv \prod_{i=1}^{\infty} P_i(M_i) . \tag{35}$$

The limit always exists because the sequence

$$\prod_{i=1}^{n} P_i(M_i) \quad (n = 1, 2, \ldots)$$

is monotonically decreasing and converges therefore strongly to a limit projection.

The spectral measure we are looking for is the extension of the set function P' defined by Equation (35) to the σ-ring of sets generated by the Borel rectangle $\underset{i}{\mathrm{X}} M_i$. One must prove by explicit construction that this extension is possible and then one must show that the extension is unique. The result is condensed in the following.

Theorem: If A_i is a countable set of commuting self-adjoint operators with spectra Λ_i and spectral measures P_i then there exists an unique spectral measure P defined on the σ-ring generated by the Borel rectangles of the product space $\underset{i}{\mathrm{X}} \Lambda_i$ and such that

$$P\,(\underset{i}{X} M_i) = \prod_{i=1}^{\infty} P_i\,(M_i) \equiv P'\,(\underset{i}{X} M_i)$$

for every Borel rectangle $\underset{i}{X} M_i$ in $\underset{i}{X} \Lambda_i$.

Outline of the proof: The extension of the projection valued set function P' can be accomplished in two phases. In the first phase we use a procedure which is a straightforward adaption of the procedure used in the analogous problem for numerical valued measures*): The projection valued set function P' is first shown to be finitely additive on the Borel rectangles. As a consequence, P' has an unique extension to a finitely additive set function P'' defined on the ring R generated by the Borel rectangles. It is then shown that P'' is not only finitely additive but also σ-additive on R.

At this point we pass to the second phase. With P'' we construct a σ-additive numerical valued totally finite measure on R by setting for an arbitrary vector $\psi \in \mathfrak{H}$.

$$\mu_\Delta(\psi) = || P''(\Delta) \psi ||^2 .$$

By a well-known extension theorem*) we can extend this numerical measure uniquely to a measure $\bar{\mu}(\psi)$ on the σ-ring $S(R)$ generated by R and such that

$$\bar{\mu}_\Delta(\psi) = \mu_\Delta(\psi)$$

for every $\Delta \in R$.

We now define a complex valued set function $\mu_\Delta(\psi, \varphi) \equiv (\psi, P''(\Delta)\varphi)$ on R which depends on an arbitrary pair (ψ and $\varphi \in \mathfrak{H}$). With the help of the process of polarization we obtain

$$\mu_\Delta(\psi, \varphi) = \frac{1}{4} \{\mu_\Delta(\psi + \varphi) - \mu_\Delta(\psi - \varphi) + i\, \mu_\Delta(\psi + i\, \varphi) - i\, \mu_\Delta(\psi - i\, \varphi)\} .$$

Every term on the right-hand side admits an unique extension to $S(R)$. There is thus one and only one σ-additive set function $\bar{\mu}_\Delta(\psi, \varphi)$ on $S(R)$ such that

$$\bar{\mu}_\Delta(\psi, \varphi) = \mu_\Delta(\psi, \varphi)$$

for every $\Delta \in R$.

One verifies easily that $\bar{\mu}_\Delta(\psi, \varphi)$ is a bounded, symmetrical bilinear functional of ψ and φ for every fixed $\Delta \in S(R)$. Therefore according to a theorem of RIESZ**) there exists for every fixed $\Delta \in S(R)$ a bounded self-adjoint linear operator $P(\Delta)$ such that

$$\bar{\mu}_\Delta(\psi, \varphi) = (\psi, P(\Delta)\, \varphi) .$$

It is now easy to verify that the operators $P(\Delta)$ are in fact projections satisfying

$$P(\Delta_1)\, P(\Delta_2) = P(\Delta_1 \cap \Delta_2)$$

The projection valued set function thus constructed is the desired measure. Its reduction to the Borel rectangles coincides with P' and its uniqueness and σ-additivity follow from the corresponding properties of the complex valued measures $\bar{\mu}_\Delta(\psi,\varphi)$. This completes the outline of the proof***).

7. *The functional calculus on the spectral measure*

Just as in the case of one operator, what interests us is a functional calculus for the von Neumann algebra generated by the operators A_i of the set $\mathfrak{S}$.

Just as in the one-dimensional case we need a theorem which we have quoted before[13]), which says that the von Neumann algebra generated by a certain spectral measure consists of all essentially bounded Borel functions of this spectral measure. But with this theorem the task is not completely finished, because the class of spectral projections P_i of the operators A_i is contained in the class of all projections of the spectral measure. We need to verify that they both generate the same von Neumann algebra. With these two hurdles overcome (Lemma 1 and 2 below) we can prove the following.

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*) Reference [14]), Theorem 13.A, p. 54.

**) Cf. reference [17]), pp. 61–62 and p. 202.

***) The complete proof is found in reference [13]).

Theorem 2: Let T be any operator of the von Neumann algebra generated by the set $\mathfrak{S}$ of commuting operators A_i. Then there exists an essentially bounded function $u(\lambda)$ defined on the product space $\underset{i}{X} \Lambda_i$ such that

$$(\psi, T\,\psi) = \int u(\lambda)\, d\mu(\psi) \text{ for all } \psi \in \mathfrak{H}\,,$$

where $\mu(\psi)$ denotes the measure defined by the equation

$$\mu_\Delta \psi) = (\psi, P(\Delta)\,\psi) \tag{36}$$

for all sets $\Delta \in S(R)$.

Lemma 1: Let P be a spectral measure defined on some σ-algebra S of subsets of a given set Λ, and let B denote the class of all spectral projections i.e. projections $P(\Delta)$ with $\Delta \in S$. Then for every $T \in B''$ there exists an essentially bounded function $u(\lambda)$ $(\lambda \in \Lambda)$ such that

$$(\psi, T\,\psi) = \int u(\lambda)\, d\mu(\psi) \text{ for all } \psi \in \mathfrak{H}\,,$$

where $\mu(\psi)$ is the measure defined by the equation

$$\mu_\Delta(\psi) = (\psi, P(\Delta)\,\psi)\,.$$

In other words T is a bounded function (in the sense of the functional calculus) of the spectral measure P.

We shall omit a formal proof of this lemma since it can be obtained by an obvious adaption of the proof given in reference [13]).

Lemma 2: The von Neumann algebra generated by the set $\mathfrak{P}$ of all projection operators which are the values of the spectral measure P is identical with the von Neumann algebra generated by the set $\mathfrak{S}$ of operators.

Proof of Lemma 2: We want to prove $\mathfrak{P}'' = \mathfrak{S}''$. Since the spectral projections of all the operators in $\mathfrak{S} \subseteq \mathfrak{P}$, it is trivial that $\mathfrak{S}'' \subseteq \mathfrak{P}''$. Thus we prove $\mathfrak{P}'' \subseteq \mathfrak{S}'$. Let $\mathfrak{P}_1$ denote the set of all spectral projections of all operators in $\mathfrak{S}$. By definition of the commutant we have $\mathfrak{P}_1' = \mathfrak{S}'$. Let $T \in \mathfrak{P}_1'$ and $T^* = T$. We shall show first that $T \in \mathfrak{P}'$.

Let Δ be any subset in the σ-ring $S(R)$ generated by the class of all Borel rectangles in $\underset{i}{X} \Lambda_i$. Then we have

$$(\psi, P(\Delta)\, T\,\varphi) = \bar{\mu}_\Delta(\psi, T\,\varphi)\,.$$

But $\bar{\mu}_\Delta(\psi, T\,\varphi)$ is the unique extension of the set function (cf. Theorem 1)

$$\mu_{\underset{i}{X} M_i}(\psi, T\,\varphi) \equiv (\psi, \prod_i P_i(M_i)\, T\,\varphi)$$

defined on the Borel rectangles $\underset{i}{X} M_i$. Since T commutes with every $P_i(M_i)$ $(i = 1, 2, \ldots)$ by assumption, we have

$$\mu_{\underset{i}{X} M_i}(\psi, T\,\varphi) = (\psi, T \prod_i P_i(M_i)\,\varphi) = (T\,\psi, P_i(M_i)\,\varphi) = \mu_{\underset{i}{X} M_i}(T\,\psi, \varphi)\,.$$

We denote, as always, the unique extension of $\mu(T\,\psi, \varphi)$ to $S(R)$ by $\mu(T\,\psi, \varphi)$. It follows from the previous equality and the uniqueness of the extension, that

$$(\psi, P(\Delta)\, T\, \varphi) = \overline{\mu}_\Delta\, (\psi, T\, \varphi) = \overline{\mu}_\Delta\, (T\, \psi, \varphi) = (\psi, T\, P(\Delta)\, \varphi)$$

for arbitrary ψ and φ in $\mathfrak{H}$ and any $\Delta \in S(R)$.
In formulae

$$T\, P(\Delta) = P(\Delta)\, T \text{ or } T \in P'$$

Since the self-adjoint elements in a von Neumann algebra generate that algebra we have proved

$$\mathfrak{P}'_1 \subseteq \mathfrak{P}'$$

consequently

$$\mathfrak{P}'' \subseteq \mathfrak{P}_1'' = \mathfrak{S}'' \qquad \text{q.e.d.}$$

The proof of theorem 2 results immediately from the combination of Lemma 1 and 2.

8. *The Spectral representation for a sequence of operators*

The ground is now prepared for establishing the spectral representation of a set $\mathfrak{S} = \{A_i\}$ $(i = 1, 2, \ldots)$ of self-adjoint operators. The program is again as in the one-dimensional case contained in the following four steps

(1) Establish a measure class ϱ defined on the σ-ring $S(R)$.
(2) Establish a bijective mapping of the abstract Hilbert space onto the space $L^2_p : f \leftrightarrow u\, (\lambda_1, \lambda_2, \ldots, \lambda_i, \ldots)$.
(3) Verify linearity and isometry of this mapping.
(4) Show that if $f \leftrightarrow u\, (\lambda_1, \ldots, \lambda_i, \ldots)$ then $A_i f \leftrightarrow \lambda_i\, u\, (\lambda_1, \ldots, \lambda_i, \ldots)$ whenever f is in the domain D_{A_i} of A_i.

The steps (1), (2), and (3) present no difficulty and can be taken over from the one-dimensional case by replacing the spectral measure $E(\Delta)$ for A by the spectral measure P constructed in section (6). The cyclic vector g of the algebra $\mathfrak{S}''$ exists since $\mathfrak{S}$ is assumed to be a complete set of commuting operators. The measure ϱ is then defined by $\varrho(\Delta) = (g, P(\Delta)\, g)$. Thus there is no need to carry through the first steps again in detail. They are the same as in the one-dimensional case.

The situation is different for step (4). In part I we have shown that the representation of the operator A as multiplication operator follows from the following two facts

(a) The correspondence $u(\lambda) \leftrightarrow u(A)$ established by the functional claculus of A is multiplicative.
(b) The function λ represents in that correspondence the operator $A : \lambda \leftrightarrow A$.

In the present case the correspondence $u\, (\lambda_1, \lambda_2, \ldots) \leftrightarrow u\, (A_1, A_2, \ldots)$ established by the functional calculus of the spectral measure P is still multiplicative, as one can easily verify. Let us however examine property (b).

We note that in the one-dimensional case property (b) is in fact trivial, as it expresses essential nothing more than the spectral resolution of the operator A.

In the general case property (b) would mean the following identity

$$\int_{\Delta_i} \lambda_i\, d\left(\int_{\underset{J}{X} M_J} d\varrho\right) = \int_{\underset{r}{X}\Delta_r} \lambda_i\, d\varrho\,, \tag{37}$$

where $M_J = \Delta_J$ for $J \neq i$ and $M_i = (-\infty, \lambda_i]$.

Indeed both sides are equal to $(g, A_i g)$ evaluated once with the definition of the spectral measure of A_i and once with the functional calculus on $\underset{i}{X} \Lambda_i$ under the assumption that A_i is represented by λ_i. One can show with examples that (37) is in general false. It is therefore of interest to know under what additional assumption it is true.

Before we formulate this additional assumption we verify that even without further assumption the correspondence $f \leftrightarrow u(\lambda_1, \lambda_2, \ldots)$ has the following property: If

$$f \leftrightarrow u(\lambda_1, \lambda_2, \ldots)$$

then

$$P_i(M_i) f \leftrightarrow \chi_{M_i}(\lambda_i)\, u(\lambda_1, \lambda_2, \ldots); \quad (i = 1, 2, \ldots) .$$

Indeed the correspondence $f \leftrightarrow u(\lambda_1, \lambda_2, \ldots)$ is obtained as follows. For a fixed cyclic vector let $f \in \mathfrak{M} = \{A\, g\}$. Then there exists a $T \in \mathfrak{A}$ such that $f = T g$. The operator T defines a.e. the function $u(\lambda_1, \lambda_2, \ldots)$ such that $T = u(A_1, A_2, \ldots)$. Since the correspondence is multiplicative we have

$$P_i(M_i) f = P_i(M_i)\, T g = T\, P_i(M_i)\, g =$$

$$T\, \chi_{M_i}(A_i)\, g \leftrightarrow \chi_{M_i}(\lambda_i)\, u(\lambda_1, \lambda_2, \ldots) .$$

The additional assumption which is needed for the validity of the identity (37) is the following:

Assumption: The measure ϱ on $S(R)$ is absolutely continuous with respect to the product measures $\mu_1 \times \mu_2 \times \ldots \times \mu_i \times \ldots$ where μ_i are the measures on the Borel sets of Λ_i defined by

$$\mu_i(M_i) = \int\limits_{\underset{J}{X} M_J} d\varrho$$

$$(M_J = \Lambda_J \text{ for } J \neq i \text{ and } M_J = \Delta \text{ for } J = i) .$$

Under this assumption we can invoke the Radon-Nikodym theorem which says that there exists a measurable function $\nu(\lambda_1, \lambda_2, \ldots)$ such that the left-hand side of (37) becomes

$$\int \lambda_i\, d\left(\int\limits_{\underset{J}{X} M_J} \nu(\lambda_1, \lambda_2, \ldots\right) d(\mu_1 \times \mu_2 \times \ldots)$$

$$= \int\limits_{\underset{r}{X} \Lambda_r} \lambda_i\, \nu(\lambda_1, \lambda_2, \ldots)\, d(\mu_1 \times \mu_2 \times \ldots) \qquad \equiv \int\limits_{\underset{r}{X} \Lambda_r} \lambda_i\, d\varrho .$$

Here the first equality is obtained from Fubini's theorem*).

We summarize the result of all that precedes with the following *Theorem 3:*

Let $\mathfrak{S} = \{A_i\}$ $(i = 1, 2, \ldots)$ be a complete set of self-adjoint operators and let Λ_i and P_i denote respectively the spectrum and the spectral measure of the operators A_i.

*) Cf. reference [17]), p. 148.

Then there exists an unique class C of equivalent measures defined on the σ-ring generated by the Borel rectangles of $\underset{i}{X} \Lambda_i$. For every $\varrho \in C$ there exists a one-one isometric correspondence $f \leftrightarrow u\,(\lambda_1, \lambda_2, \ldots, \lambda_i \ldots)$ between $\mathfrak{H}$ and L^2_ϱ with the property that if

$$f \leftrightarrow u\,(\lambda_1, \lambda_2, \ldots, \lambda_i \ldots),$$

then

$$P_i(M_i)\, f \leftrightarrow \chi_{M_i}(\lambda_i)\; u\,(\lambda_1, \lambda_2, \ldots, \lambda_1 \ldots), \quad (i = 1, 2, \ldots)$$

where M_i denotes a Borel subset of Λ_i and $\chi_{Mi}(\lambda_i)$ the characteristic function of M_i.

If the measure ϱ is absolutely continuous with respect to the product measure of the measures μ_i on the Borel sets Λ_i then the correspondence has the additional property that

$$A\, f \leftrightarrow \lambda_i\, u(\lambda_1, \lambda_2, \ldots, \lambda_i \ldots)\,.$$

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Appendix

The Gelfand spectral representation (*GSR*)

A 1. *The Gelfand isomorphism of Abelian Banach algebras*[15])

Let $\mathfrak{A}$ be a commutative Banach algebra with identity and M a maximal ideal of $\mathfrak{A}$. It can be proved that every element $\tilde{x}$ of the factor ring $\mathfrak{A}/M$ is of the form

$$\tilde{x} = \lambda\, \tilde{e} \tag{A 1}$$

where λ is a complex number and $\tilde{e}$ is the identity of $\mathfrak{A}/M$. Thus the correspondence $\tilde{x} \to \lambda$ is an isomorphism of $\mathfrak{A}/M$ onto the field of complex numbers. It will be called the Gelfand-Mazur isomorphism.

The natural homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}/M$ with kernel M, followed by the Gelfand-Mazur isomorphism of $\mathfrak{A}/M$ onto the field of complex numbers C will induce a homomorphism of $\mathfrak{A}$ onto C. This will be called the natural homomorphism of $\mathfrak{A}$ generated by the maximal ideal M.

It is now easy to verify the following

Theorem: There exists a one-one correspondance $h \to M$, between the class of all homomorphisms of $\mathfrak{A}$ onto C and the set of all maximal (non-trivial) ideals in $\mathfrak{A}$, such that, if the homomorphism h corresponds to the maximal ideal M, then h is the natural homomorphism generated by M.

Let x be an element of $\mathfrak{A}$. Then $x(M)$ denotes the complex number assigned to x through the natural homomorphism generated by the maximal ideal M. For any fixed $x \in \mathfrak{A}$, we obtain in this way a complex valued function $x(M)$ defined on the set $\mathfrak{M}$ of all maximal ideals of $\mathfrak{A}$.

The set of all complex valued functions $x(M)$ over $\mathfrak{M}$ is an algebra $\mathfrak{F}(\mathfrak{M})$ if addition, scalar multiplication and multiplication of functions are defined as usual:

$$\begin{aligned} (x_1 + x_2)\,(M) &= x_1(M) + x_2(M)\,, \\ (\alpha\, x)\,(M) &= \alpha\, x(M) \text{ for complex } \alpha \\ (x_1\, x_2)\,(M) &= x_1(M)\, x_2(M)\,. \end{aligned} \tag{A2}$$

The correspondence $x \to x(M)$ is then a homomorphism of the algebra $\mathfrak{A}$ into $\mathfrak{F}(\mathfrak{M})$. One can prove that $x(M) = 0$ if and only if $x \in M$. Furthermore if M_1 and M_2 are two distinct maximal ideals of $\mathfrak{A}$ then there exists at least one $x \in \mathfrak{A}$ such that $x(M_1) \neq x(M_2)$.

The sprectrum of an element $x \in \mathfrak{A}$ is defined as the set of all the complex numbers λ such that there exists no $y \in \mathfrak{A}$ satisfying

$$y\,(x - \lambda\, e) = (x - \lambda\, e)\, y = e\,.$$

For fixed $x \in \mathfrak{A}$ the set of values assumed by the function $x(M)$ is identical with the spectrum of x. It is possible to define a topology in $\mathfrak{M}$ such that all functions $x(M)$ as x runs over the algebra become continuous functions on $\mathfrak{M}$. It can be shown that in this topology the set $\mathfrak{M}$ becomes a compact topological space which we shall denote by the same letter $\mathfrak{M}$. It is called the *carrier space* of the Banach algebra $\mathfrak{A}$. The correspondence $x \to x(M)$ is now a homomorphism of the algebra $\mathfrak{A}$ into the algebra $C(\mathfrak{M})$ of all continuous functions on the carrier space.

For an important special case which interests us especially, this homomorphism becomes an isomorphism. This is always then the case if there exists an involution $x \to x^*$ with the properties

$$\begin{aligned} (\lambda\, x + \mu\, y)^* &= \lambda^*\, x^* + \mu^*\, y^* \\ (x^*)^* &= x \\ (xy)^* &= y^* x^* \\ \|\, xx^* \,\| &= \|\, x \,\|^2\,. \end{aligned} \tag{A3}$$

Such an algebra is called a commutative C^*-algebra. We have the following

Theorem: Let $\mathfrak{A}$ be a commutative C^*-algebra with identity. Then the correspondence $x \to x(M)$ is one-one and maps $\mathfrak{A}$ onto $C(\mathfrak{M})$. Furthermore it satisfies the conditions:

$$\begin{aligned} &\text{(a)} \quad \lambda x + \mu y \to \lambda x(M) + \mu y(M)\,, \\ &\text{(b)} \quad xy \to x(M)\, y(M)\,, \\ &\text{(c)} \quad x^* \to x(M)^* \quad \text{(complex conjugation)}\,, \\ &\text{(d)} \quad \|\, x \,\| = \sup_{M \in \mathfrak{M}} |\, x(M) \,|\,. \end{aligned} \tag{A4}$$

This is called the Gelfand isomorphism of the C^*-algebra $\mathfrak{A}$ onto the $*$-Banach algebra $C(\mathfrak{M})$.

A2. *The Gelfand spectral representation*

We shall now sketch how the preceding facts can be used for obtaining a spectral representation for a complete set of bounded commuting and self-adjoint operators.

Let $\mathfrak{S} = \{A_1, A_2, \ldots, A_i, \ldots\}$ be such a complete set, such that

$$\mathfrak{S}' = \mathfrak{S}'' = \mathfrak{A} .$$

Let $\mathfrak{A}_U$ denote the smallest algebra of bounded operators (with the identity operator) which is closed under the uniform topology and which contains the set $\mathfrak{S}$. When the norm on the *-algebra $\mathfrak{A}_U$ is identified with the operator norm, it can be considered as a commutative C^*-algebra with identity.

Hence by the preceding theorem $\mathfrak{A}_U$ is isomorphic to the algebra $C(\mathfrak{M})$ of all continuous functions on the carrier space $\mathfrak{M}$ of $\mathfrak{A}_U$.

In order to obtain a spectral representation of the operators in $\mathfrak{S}$ we need cyclic vectors. The existence of cyclic vectors for $\mathfrak{S}$ is guaranteed because $\mathfrak{A}$ is a maximal abelian von Neumann algebra*). The following lemma assures the existence of cyclic vectors for $\mathfrak{A}_U$ as well:

Lemma: A vector $\psi \in \mathfrak{H}$ is cyclic for $\mathfrak{A}_U$ if and only if it is cyclic for $\mathfrak{A}$.

Proof: The inclusions $\mathfrak{S} \subset \mathfrak{A}_U \subset \mathfrak{A}$,
imply

$$\mathfrak{A}' \subseteq \mathfrak{A}_U' \subseteq \mathfrak{S}' \equiv \mathfrak{A}' .$$

Therefore

$$\mathfrak{A}_U' = \mathfrak{A}' .$$

Now it is known that a vector ψ is cyclic for a *-algebra $\mathfrak{R}$ containing the identity if and only if it is separating**) for the commutant $\mathfrak{R}'$.

Let then ψ be cyclic for $\mathfrak{A}$ and denote with Q the projection with range $\{\mathfrak{A}_U \psi\}$. One can show that $Q \in \mathfrak{A}' = \mathfrak{A}_U'$. Furthermore $Q \psi = \psi$, or $(I - Q) \psi = 0$, $I - Q \in \mathfrak{A}'$ and ψ is separating for $\mathfrak{A}'$. This means $I - Q = 0$, or $Q = I$. This is precisely the statement that ψ is cyclic for $\mathfrak{A}_U$.

If conversely ψ is cyclic for $\mathfrak{A}_U$ it is all the more so for $\mathfrak{A}$ since $\mathfrak{A}_U \subset \mathfrak{A}$. This proves the lemma.

Every cyclic vector ψ for $\mathfrak{A}_U$ defines a linear functional $f(T)$ on $\mathfrak{A}_U$ by setting

$$f(T) = (\psi, T \psi) . \tag{A6}$$

In view of the Gelfand isomorphism of $\mathfrak{A}_U$ onto the set $C(\mathfrak{M})$ of all continuous functions on the carrier space $\mathfrak{M}$ of $\mathfrak{A}_U$ this linear functional may be considered as a functional on $C(\mathfrak{M})$. In fact if the continuous function $T(M)$ corresponds under the Gelfand isomorphism to the operator $T \in \mathfrak{A}_U$ we define

$$f\{T(M)\} = (\psi, T \psi) .$$

*) Cf. reference [10]).

**) Cf. reference [11]), p. 6. A vector ψ is called a separating vector of an algebra $\mathfrak{R}$ if $T \psi = 0$ and $T \in \mathfrak{R}$ implies $T = 0$.

The functional f, thus defined on $C(\mathfrak{M})$ has the following properties:

(a) $f\{T_1(M) + T_2(M)\} = f\{T_1(M)\} + f\{T_2(M)\}$,
(b) $f\{\lambda\, T(M)\} = \lambda f\{T(M)\}$,
(c) $|f\{T(M)\}| \equiv |(\psi, T\psi)| \leqslant \|T\|^2 \|\psi\|^2$,
(d) $f\{T(M)\} \geqslant 0$ *if* $T(M) \geqslant 0$ for all M.

Thus f is a bounded linear functional on $C(\mathfrak{M})$ and according to a well-known theorem*) of measure theory there exists a measure defined on the Borel sets of $\mathfrak{M}$ such that

$$f\{T(M)\} = \int_{\mathfrak{M}} T(M)\, d\,\mu(M) \tag{A 7}$$

for all $T(M)$.

The measure μ will in general depend on the cyclic vector ψ, but one can prove that two measures obtained from two different cyclic vectors are equivalent.

We can now define a correspondence between every $f \in \{\mathfrak{A}_U\, \psi\}$ and the elements of the dense linear subset $C(\mathfrak{M})$ of $L^2_\mu(\mathfrak{M})$ by the assignment

$$T\,\psi \to T(M)\,. \tag{A 8}$$

Since

$$\|T\,\psi\|^2 = (T\,\psi, T\,\psi) = (\psi, T^*\, T\,\psi) = \int_{\mathfrak{M}} |T(M)|^2\, d\mu\,,$$

this correspondence is norm preserving. We can therefore extend it in the standard manner by continuity to an unitary mapping of $\mathfrak{H}$ onto $L^2(\mathfrak{M})$.

If a vector $\phi \in \mathfrak{H}$ goes under this mapping to the function $x(M) \in L^2(\mathfrak{M})$ then the vector $A_i\,\phi$ goes to the function $A_i(M)\, x(M)$, where $A_i(M)$ is the function which is assigned to the operator $A_i(M)$ by the Gelfand isomorphism.

With this prodedure we have obtained a representation of the space $\mathfrak{H}$ such that the operators A_i appear as multiplication operators by the functions $A_i(M)$. This is called the Gelfand spectral representation (GSR).

If we compare the *GSR* with the spectral representation established in this paper we observe the following differences:

(1) The *GSR* can only be established for bounded operators, while we have shown the existence of a spectral representation even for the unbounded case.
(2) In the *GSR* the functions are defined in a carrier space $\mathfrak{M}$ with a particular topology so that the elements $T \in \mathfrak{A}_U$ become continuous functions on $\mathfrak{M}$.
(3) The carrier space $\mathfrak{M}$ depends in a not very transparent way on the set $\mathfrak{S}$ of bounded operators. In general different operator sets $\mathfrak{S}$ define different algebras $\mathfrak{A}_U$. The connection between the carrier space and the topological product space $\underset{i}{\mathrm{X}}\, \Lambda_i$ is brought to light in the following

*) Cf. reference [17]), chapter III.

Theorem: The mapping $M \to \{A_i(M), \ldots, A_i(M), \ldots\}$ with $M \in \mathfrak{M}$ is a topological homeomorphism of the space $\mathfrak{M}$ onto a subset of the Cartesian product space of the spectra of the operators $A_i \in \mathfrak{S}$.

We shall not prove this theorem here as it is a simple corollary of a well-known theorem[16]. In general this homeomorphism maps $\mathfrak{M}$ onto a proper subset of the topological space of the Λ_i.

The measure μ on the Borel sets of $\mathfrak{M}$ will induce a measure on the Borel sets of the image of $\mathfrak{M}$. In case the image of $\mathfrak{M}$ is the entire product space $\underset{i}{X} \Lambda_i$ it seems likely that this measure is equivalent to the measure which we have constructed in the main part of the paper. We have however not verified this.

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