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by **Derek W. Robinson**
Institut für Theoretische Physik, ETH, Zürich

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Abstract. The zero-mass representations of the inhomogeneous Lorentz group are considered as contractions of the representations for real and imaginary masses. It is shown that all zero-mass representations may be obtained from contracting either real mass or imaginary mass representations, but that it is more reasonable to associate the representations describing zero-mass particles with infinite angular momentum with contractions of the imaginary mass representations.

1. Introduction

A basic axiom of all present day relativistic quantum theories is the invariance of the theory under transformations of the inhomogeneous Lorentz group. Thus all physical systems are assumed to furnish a representation of this group, and it is important to understand the nature of these representations to realise the implications of relativistic invariance. The irreducible representations have been often studied and classified and the results of this classification are well known. The group has two invariants and each irreducible representation may be classified by the value of these invariants. This will be discussed further in the following sections but we briefly note that the value of one of the invariants may be interpreted as the mass of the physical system represented. Representations have been found for real, imaginary, or zero mass. The purpose of this paper is to examine the connection between zero-mass representations and representations for non-zero mass.

In sections 2 and 3 a short summary is given of the theory of the inhomogeneous Lorentz group and the classification of its representations. Section 4 provides a précis of the theory of group contraction and section 5 summarises the representations of the 'little groups'. In section 6 it is shown how the various 'little groups' contract and in which way the representations of these groups may be contracted. Conclusions concerning the contractions of the full group are given in Section 7.

2. Inhomogeneous Lorentz group

An inhomogeneous Lorentz transformation is defined as the product operation of a translation by a real vector a_μ and a homogeneous Lorentz transformation with real coefficients Λ^ν_μ. This may be written

$$x'_\mu = (L\, x)_\mu = \Lambda^\nu_\mu\, x_\nu + a_\mu\,.$$

The translation is performed after the homogeneous transformation, and with this understanding the formula for the product of two transformations is given by

$$\{a_1, \Lambda_1\}\,\{a_2, \Lambda_2\} = \{a_1 + \Lambda_1\, a_2, \Lambda_1 \Lambda_2\}\,.$$

The homogeneous transformations are restricted to the proper Lorentz group by the following conditions:

1. The transformations leave the fundamental indefinite quadratic form $g_{\mu\nu}\, x^\mu\, x^\nu$ invariant. (The metric $g_{00} = 1\; g_{11} = -1\; g_{22} = -1\; g_{33} = -1$ is used).
2. $\Lambda^0_0 > 0$ i.e. the transformations do not interchange past and future.
3. Det $|\Lambda^\mu_\nu| = +1$ i.e. this and condition 2. ensure that transformations preserve the distinction between right- and left-handed coordinate axes.

In quantum theory it is required to find continuous unitary representations (up to a factor), $\mathfrak{D}(L)$ of the inhomogeneous group. The operator multiplication rule is given by

$$\mathfrak{D}(L_1)\; \mathfrak{D}(L_2) = \omega(L_1, L_2)\; \mathfrak{D}(L_1, L_2)$$

where $\omega(L_1, L_1)$ is a number of modulus one. This relation may be simplified by permissible phase changes of the representation, and the work of Wigner and Bargmann has shown that all representations may be converted into the representations having

$$\omega(L_1, L_2) = \pm 1\,.$$

Transformations of particular importance are the infinitesimal Lorentz transformations, which may be written as

$$\Lambda^\nu_\mu = g^\nu_\mu + \epsilon^\nu_\mu\,,$$

$$a_\mu = \epsilon_\mu$$

where $\in_\mu^\nu$ and $\in_\mu$ are infinitesimally small. Condition (1) then leads to the restriction

$$\in_{\mu\nu} + \in_{\nu\mu} = 0 .$$

We now define the infinitesimal generators of the group P_μ and $M_{\mu\nu}$ through the representation of the infinitesimal transformation

$$\mathfrak{D}(L) = 1 + \frac{1}{2}\, i\, M^{\mu\nu} \in_{\mu\nu} + i\, P^\mu \in_\mu .$$

This equation defines ten operators P^μ and $M^{\mu\nu}$ ($M^{\mu\nu} = - M^{\nu\mu}$). The Hermitian operators P^μ are the generators for infinitesimal translations and represent the linear momentum of the system, whilst the Hermitian operators $M_{\mu\nu}$ are the generators for infinitesimal 'rotations' in the $x^\mu - x^\nu$ plane. The commutation relations of these operators follow from the theory of Lie groups and are given by the well known relations

$$\begin{aligned} [P^\mu, P^\nu] &= 0 , \\ [M^{\mu\nu}, P^\lambda] &= i\, (P^\nu g^{\mu\lambda} - P^\mu g^{\nu\lambda}) , \\ [M^{\mu\nu}, M^{\varrho\sigma}] &= i\, (g^{\mu\sigma} M^{\nu\varrho} - g^{\mu\varrho} M^{\nu\sigma} + g^{\nu\varrho} M^{\mu\sigma} - g^{\nu\sigma} M^{\mu\varrho}) . \end{aligned} \tag{1}$$

It is worthwhile to comment that as the Lorentz group is a non-compact Lie group the validity of using the classical methods of infinitesimal generators is not at all obvious. The mathematical justification of such methods has been given by GÅRDING[2]) although these techniques were used much earlier by several workers.

In order to find the invariants of the group in terms of the above operators we define the new operator

$$\omega_\sigma = \frac{1}{2} \in_{\sigma\mu\nu\lambda} M^{\mu\nu} P^\lambda \tag{2}$$

where $\in_{\sigma\mu\nu\lambda}$ is an operator completely antisymmetric in its indices and with the normalization property

$$\in_{0123} = 1 .$$

The operator ω_σ satisfies the relation

$$\omega_\mu P^\mu = 0 \tag{3}$$

and also the commutation relations

$$\begin{aligned} [\omega_\mu, P_\nu] &= 0 , \\ [\omega_\varrho, M_{\mu\nu}] &= i\, (g_{\mu\varrho}\, \omega_\nu - g_{\nu\varrho}\, \omega_\mu) , \\ [\omega_\mu, \omega_\nu] &= i \in_{\mu\nu\varrho\sigma} \omega^\varrho P^\sigma . \end{aligned} \tag{4}$$

It is easily proved that the scalar operators P and W, defined by

$$
\begin{aligned}
P &= P_\mu P^\mu , \\
W &= - \omega_\mu \omega^\mu , \qquad\qquad (5) \\
&= \frac{1}{2} M_{\mu\nu} M^{\mu\nu} P_\sigma P^\sigma - M_{\mu\sigma} M^{\nu\sigma} P^\mu P_\nu
\end{aligned}
$$

commute with all the infinitesimal operators, $M_{\mu\nu}$, P_μ of the group. Therefore, for every irreducible representation of the group the operators P and W are multiples of the identity. It now follows that the irreducible representations of the group may be classified by the eigenvalues of P and W and a complete set of commuting operators may be chosen from the P_μ and ω_μ. Of course many different sets may be chosen, all of them giving rise to equivalent representations, however in this work we choose the set $(P_0\, P_1\, P_2\, P_3\, L_0)$ where L_0 is a certain linear combination of the operators ω_μ which will be defined in the next section. The eigenvalue spectrum of these operators then specifies the range of variables labeling the basis vectors.

The physical interpretation of the invariant P is clearly the square of the physical mass. The interpretation of W can be made as follows. The operator $M_{\mu\nu}$ is decomposed into two parts

$$M_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu} \qquad (6)$$

where $L_{\mu\nu}$ is an operator acting in the space of the momentum variables only and $S_{\mu\nu}$ acts in the space of the remaining variable. Then

$$L_{\mu\nu} = i \left(P_\mu \frac{\partial}{\partial P^\nu} - P_\nu \frac{\partial}{\partial P^\mu} \right) \qquad (7)$$

and this satisfies the equation

$$\frac{1}{2} \epsilon_{\sigma\mu\nu\lambda} L^{\mu\nu} P^\lambda = 0$$

and hence

$$\omega_\sigma = \frac{1}{2} \epsilon_{\sigma\mu\nu\lambda} S^{\mu\nu} P^\lambda . \qquad (8)$$

Now in the rest frame of a particle of real mass m the operator W takes the form

$$W = m^2 S_{ij} S^{ij} , \qquad i, j = 1, 2, 3 \qquad (9)$$

i.e. W is the product of the square of the mass and the intrinsic angular momentum of the particle.

3. Classification of representations

In order to classify the irreducible unitary representations of the inhomogeneous Lorentz group we consider more closely the operators ω_μ. These operators are the infinitesimal generators of the group which leaves the linear momentum P_μ invariant. It follows from the relation (2. 3) that only three of the four operators ω_μ are independent. (We omit from our considerations the case $P_\mu \equiv 0$) so that this group is a three parameter group and W is the invariant of the group. This group is the 'little group' of WIGNER[1]). The representations of the inhomogeneous Lorentz group are determined by the representations L_ϱ of the little group and a measure on momentum space. The irreducible representations can now be divided into four classes as follows

$$\begin{array}{llll} 1.\ P_\mu P^\mu = P > 0\,, & & \text{a)}\ P_0 > 0\,, & \text{b)}\ P_0 < 0\,, \\ 2.\ P_\mu P^\mu = P < 0\,, & & & \\ 3.\ P_\mu P^\mu = P = 0\,, & P_\mu \neq 0\,, & \text{a)}\ P_0 > 0\,, & \text{b)}\ P_0 < 0\,, \\ 4.\ P_\mu = 0\,, & & & \end{array} \tag{1}$$

We consider the first three of these classes in which the variability domain of P_μ is three dimensional. As ω_μ is orthogonal to P_μ we may express it in the following manner:

$$\omega_\mu = L_i\, n^i_\mu\,, \qquad i = 0, 1, 2 \tag{2}$$

where n^i_μ are a set of orthogonal vectors spanning the space orthogonal to P_μ. The operators L_i are three independent generators of the 'little group' and to complete their definition we must specify the character and normalization of the vectors n^i_μ. In all three classes two of the vectors n^i_μ must be space-like and the third vector is either space-like, time-like or parallel to P_μ corresponding respectiv.ly to classes 1., 2. and 3. Thus we choose

$$\begin{aligned} n^0_\mu\, n^{0\mu} &= -P\,, \\ n^i_\mu\, n^{i\mu} &= -1\,, \qquad i = 1, 2\,. \end{aligned} \tag{3}$$

It now follows from the commutation relations of the ω_μ (2.4) and the definition of W (2.5) that the operators L_i satisfy the commutation relations

$$[L_i, L_j] = i\, \epsilon_{ijk}\, g^{kl}\, L_l \tag{4}$$

and that

$$W = g^{ij} L_i L_j , \tag{5}$$

In these formulae ϵ_{ijk} is an operator completely antisymmetric in its indices with the normalization property

$$\epsilon_{012} = 1$$

and g^{ij} is the metric tensor defined by $g^{00} = P$ $g^{11} = 1$ $g^{22} = 1$ and with all other components zero.

The explicit values of the components of the metric tensor depend upon the choice of the vectors n^i_μ. The magnitude of the components of this tensor depends upon the magnitude of the normalization of the vectors n^i_μ and the signs depend upon the nature of the vectors. The nature of the vectors n^i_μ is completely determined by the nature of P_μ, and thus the signs of the metric tensor components are determined by the nature of the linear momentum vector P_μ, and these signs in turn determine the character of the 'little group' generated by L_i. The above choice of normalization is chosen in order to simplify the algebra occurring in the following sections. Any variation of the momentum vector P_μ which leads to a change in the value of P results in a change in the group generated by the L_i. If however P varies over any closed interval on the real line which excludes the point zero the group only undergoes a series of isomorphic transformations.

We wish to examine the limiting behaviour of the series of representations of the group for non-vanishing P in the limit that P approaches zero. It is in this limit that the nature of the vector n^0_μ changes and the metric tensor g^{ij} degenerates. This type of limit of a group has been first studied in quite general mathematical form by SEGEL[4]) and later in less generality by WIGNER and INÖNÜ[5]), and also SALETAN[6]). These latter authors consider some physical applications of the theory. As the theory is not too well known we reproduce in the next section some of the methods and results; we follow the appproach of WIGNER and INÖNÜ who name the process group contraction.

4. Group Contraction

Contraction of a Lie group is defined by WIGNER and INÖNÜ in the following sense. Consider an arbitrary Lie group G with n infinitesimal operators I_i and structure constants C^k_{ij} defined by

$$[I_i, I_j] = C^k_{ij} I_k . \tag{1}$$

Let the infinitesimal generators I_i be subjected to a linear homogeneous non-singular transformation denoted by

$$J_r = U_r^i \, I_i \, . \tag{2}$$

This transformation is an isomorphism of the group upon itself and may only lead to a new group if U is singular. We consider the possibility of this happening by assuming U to have the form

$$U = u + P\, w \tag{3}$$

where in terms of sub-matrices

$$u = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad w = \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}$$

and P is a constant lying in some range $0 < P < P_0$ for which U is non-singular. This transformation of the infinitesimal elements will also change the structure constants of the group and we find that if

$$[J_r, J_s] = \overline{C}^t_{rs} \, J_t$$

then

$$\overline{C}^t_{rs} = U_r^i \, U_s^j \, C^k_{ij} \, (U^{-1})^t_k \, . \tag{4}$$

If q is the rank of u it is now easily shown from (4.3) and (4.4) that

$$\overline{C}^t_{rs} = \left[\frac{1}{P} + O(P^0)\right] C^t_{rs} + O(P) \, , \quad r, s \leqq q \, , \quad t > q \, ,$$
$$\overline{C}^t_{rs} = P\, C^t_{rs} \, , \qquad r, s > q$$

and that all other structure constants $\overline{C}^t_{rs}$ are of order P^0 or higher power in P. Hence, if in the limit $P \to 0$ the commutator of J_r and J_s is to converge to a linear combination of the J, then we must have

$$C^t_{rs} = 0 \, , \quad r, s \leqq q \, , \quad t > q \tag{5}$$

or alternatively, the operators I_r for $r \leqq q$ must span a subgroup S of G. This operation is defined as the contraction of G with respect to S. Wigner and Inönü also express this transition by stating that the infinitesimal elements J_r for $r < q$ are contracted. It follows from (4.5) that the contracted infinitesimal elements form an Abelian invariant subgroup of the contracted group.

If one applies the transformation (4.2) to the infinitesimal elements of a representation D of the group to be contracted, and lets P tend to zero, the J_r $(r > p)$ will also tend to zero. Thus the representation obtained by this method is never faithful. INÖNÜ and WIGNER suggest two ways in which faithful representations of the contracted group may be obtained from the representations D of the group to be contracted. The first method is to perform a P – dependent similarity transformation on the representation D, the second is to consider a sequence of representations which converges to a representation of the contracted group as P tends to zero. The second method will be used in the following sections and the definition of the type of convergence involved will be discussed in relation to the contractions of interest. Further details of the general theory of group contraction may be found in the references quoted above. We now turn our attention to an examination of the 'little group', and its representations, in the three classes of representations under consideration.

5. The little Groups

The characters of the little groups as defined in section 3 are easy to analyse, and the unitary representations of these groups are all known. We give in this section a brief discussion of each of the three little groups and a summary of the unitary representations. The section is split into three parts corresponding to the three classes of representations of the complete group.

Class 1: In this class $P > 0$ and the little group is isomorphic to the three dimensional rotation group. As the group is compact the only irreducible unitary representations are of finite dimension. These representations, D_l, are of dimension $2l + 1$, where l is either integer or half integer, and the corresponding eigenvalues of are given by

$$W = P\,l\,(l + 1)\,, \qquad l = 0, \frac{1}{2}, 1 \ldots \tag{1}$$

From the discussion at the end of section (2) we see that l may be interpreted physically as the spin of a particle of physical mass $\sqrt{P}$.

The only non-vanishing matrix elements of the operators L_i in these representations are given by

$$\begin{gathered} \langle l\,m \mid L_0 \mid l\,m\rangle = m\,, \\ \langle l\,m+1 \mid L_+ \mid l\,m\rangle = \langle l\,m \mid L_- \mid l\,m+1\rangle = [P\,(l-m)\,(l+m+1)]^{1/2} \end{gathered} \tag{2}$$

where $L_\pm$ are defined by

$$L\,\pm = L_1 \pm i\,L_2\,.$$

The dimensionality of the representation is evident from the observation that

$$L_+ \,|\, l, l\rangle = 0 = L_- \,|\, l, -l\rangle$$

and so in each representation the values of m occurring are

$$-l, -l+1, \ldots, \qquad l-1, l\,.$$

Class 2: In this class $P < 0$ which corresponds to space-like linear momentum. The little group is isomorphic to the three-dimensional homogeneous Lorentz group. The irreducible unitary representations of this group*) have been given by BARGMANN[7]), who showed that they fall into two separate classes, the 'discrete class' and the 'continuous class'. These names derive from the nature of the spectrum of the operator W. We discuss first the 'discrete class' of representations.

If l is again a positive integer or half integer there is a unitary representation for each l and the appropriate eigenvalue of W is given by

$$W = -\,P\,l\,(1-l)\,. \tag{3}$$

The only non-vanishing matrix elements of the operators L_i in these representations are given by

$$\begin{gathered}\langle l\,m \,|\, L_0 \,|\, l\,m\rangle = m\,,\\ \langle l\,m+1 \,|\, L_+ \,|\, l\,m\rangle = \langle l\,m \,|\, L_- \,|\, l\,m+1\rangle = [-P(l+m)\,(m-l+1)]^{1/2}.\end{gathered} \tag{4}$$

Observing that

$$L_+ \,|\, l, -l\rangle = 0 = L_- \,|\, l\,l\rangle\,,$$

it is seen that the representation splits into two infinite dimensional irreducible representations D_l^+ and D_l^-. The representation D_l^+ contains positive values of m i.e. $m = l,\ l+1,\ l+2 \ldots$ and the representation D_l^- contains negative values of m i.e. $m = -\,l,\ -l-1,\ -l-2\ldots.$

The second class of unitary representations the 'continuous class' is derived from the assumption that no state vector satisfies either of the equations

$$L_\pm \,|\, l\,m\rangle = 0\,.$$

Hence the representations are again infinite dimensional. The eigenvalue spectrum of W is found to be continuous positive i.e.

$$W = -\,P\,q \tag{5}$$

where q is a positive constant.

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*) We consider only the single-valued and double valued representations (see BARGMANN, refs. [1]) and [7])).

The only non-vanishing matrix elements of the operators L_i in the representation corresponding to a chosen eigenvalue of W are given by

$$\begin{aligned} \langle q\, m \mid L_0 \mid q\, m \rangle &= m\,, \\ \langle q\, m+1 \mid L_+ \mid q\, m \rangle = \langle q\, m \mid L_- \mid q\, m+1 \rangle &= [-P\{q+m\,(m+1)\}]^{1/2} \end{aligned} \tag{6}$$

where all vectors are normalized to unity.

If m assumes integer values the representations (C_q^1) are unitary for all values of $q > 0$. However, the representations $C_q^{1/2}$ in which m assumes half-integer are unitary only if $q \geqq 1/4$. In the interval $0 < q \leqq 1/4$ unitary representations may be constructed, but as the properties of this exceptional interval are of no direct interest to us we omit details of these representations.

Class 3: In this class $P = 0$ and the little group is the two dimensional Euclidean group. The operator W is positive semidefinite

$$W = L_1^2 + L_2^2 = L_+ L_-\,,$$

and the unitary representations are determined by

$$W = k^2 \tag{7}$$

and

$$\begin{aligned} \langle k\, m \mid L_0 \mid k\, m \rangle &= m\,, \\ \langle k\, m+1 \mid L_+ \mid k\, m \rangle = \langle k\, m \mid L_- \mid k\, m+1 \rangle &= k\,. \end{aligned} \tag{8}$$

The nature of these representations depends radically on whether k is zero or non zero.

a) If $K = 0$ then $W = 0$ and the operators L_1, and L_2 also vanish. Therefore the representations, E_0^m, are all one dimensional. This representation is a non-faithful representation of the group.

b) If $k \neq 0$ then the representations are all infinite dimensional faithful representations, E_k.

In order to establish the physical meaning of these representations we consider the definition of the L_i given by (3.2)

$$\omega_\mu = L_0 P_\mu + L_1 n_\mu^1 + L_2 n_\mu^2\,.$$

We now choose the vectors n_μ^i such that $n_0^1 = 0 = n_0^2$. This special choice of n_μ^1 and n_μ^2 is always possible. Then we have from (2.8) that

$$\begin{aligned} \omega_0 &= \frac{1}{2} \epsilon_{0\mu\nu\lambda}\, S^{\mu\nu} P^\lambda \\ &= \boldsymbol{S} \cdot \boldsymbol{P} \end{aligned}$$

where $\boldsymbol{S}$ is the spin vector defined by

$$S_i = \epsilon_{ijk} S^{ik}, \qquad i, j, k = 1, 2, 3 .$$

Therefore

$$L_0 = \frac{\omega_0}{P_0} = \frac{\boldsymbol{S} \cdot \boldsymbol{P}}{|\boldsymbol{P}|} .$$

This operator is usually called the helicity operator and it represents the component of spin of a particle along its direction of motion.

The one-dimensional representations of the group are therefore representations of a particle with zero mass and with one value of the helicity e.g. the neutrino. If in the physical theory a parity operation is defined then a particle should have two directions of helicity $\pm \lambda$, because $\boldsymbol{P}$ is a pseudoscalar, e.g. a photon. The infinite dimensional representations would however describe a completely different situation. The possible values of the helicity of the particle are again given by m, but each representation allows all integer, or half-integer, values for m as opposed to the one-dimensional representations which allow only a unique values. Thus the physical situation described by these representations would be a zero-mass particle with an infinite number of helicity states and infinite angular momentum.

The purpose of the following section is to show in which manner the representations of class 3 are contractions of representations of class 1 and class 2.

6. Contractions of the little Groups

The little group of class I, as defined in sections 3 and 5, is isomorphic to the three-dimensional rotation group so long as $P > 0$. The limit $P \to 0$ contracts the group with respect to rotations about the x_0-axis and the resultant contracted group is the two-dimensional Euclidean group i.e. the little group of class 3.

Simple contraction of the representations of the group i.e. the simple limit $P \to 0$, splits the representation into $2l+1$ one-dimensional unfaithful representations of the two-dimensional Euclidean group E_0^m, as can be easily verified from (5.2) and (5.8). In order to obtain faithful representations of the contracted group we must use the method mentioned in section 4 of considering a sequence of representations which converges to a representation of the contracted group as P tends to zero. The convergence that we are interested in is the convergence of the infinitesimal operators L_i, whose matrix elements in the representation D_l are given by (5.2). We write, with an obvious notation, $D_{lm}(L_i(P))$ for these matrix elements and $E_{km}(L_i)$ for the matrix elements of the contracted group given by (5.8). Now for any integer (half integer) m there exists an integer (half integer) value of l, which we call L such that

$$P, \in > 0; \qquad |D_{lm}(L_i(P)) - E_{km}(L_i)| < \in .$$

It is clearly seen that for small P the value of L is bounded below by a number inversely proportional to the square root of P. Thus

$$\lim_{L \to \infty} \lim_{P \to 0} D_{lm}(L_i(P)) = E_{km}(L_i)$$

if the limits are understood in the sense that the limit on P is continuous whilst the limit on L is stepwise and taken in such a way that

$$\lim_{L \to \infty} \lim_{P \to 0} L^2 P = k^2 .$$

In this sense the sequence of representations

$$D_{l_0}, D_{l_0+1}, \ldots, D_{l_0+n} \cdots$$

converge to the representations E_k. The alternative method of obtaining faithful representations i.e. applying a P-dependent similarity transformation, has no application here as the method preserves the dimensionality of the representation whilst in this contraction it is essential to form the transition from finite dimensional representations to infinite dimensional representations. The method would only have application if the infinitesimal generators are not bounded operators, which means for irreducible representations that the group is not compact. In order to obtain faithful representations of the contracted group, which is necessarily non-compact, when the original group is compact the rather artificial limiting process of a sequence of finite dimensional representations, as illustrated above, must be used. This contraction has been considered more fully by WIGNER and INÖNÜ[5]) and the significance of the contraction in the context of the present work has been noted by V. I. RITUS[8]).

We now turn our attention to the little group of class 2. This group as defined previously is isomorphic to the three-dimensional homogeneous Lorentz group as long as $P < 0$. The limit $P \to 0$ contracts the group with respect to rotations about the x_0-axis i.e. rotations in the two space dimensions. It can now be shown that all representations for which $W < 0$ cannot be contracted into faithful representations of the contracted group. The argument is as follows. We have

$$W = P L_0^2 + L_1^2 + L_2^2 = -k < 0$$

therefore

$$\sqrt{-P}\, L_0 = (k + L_1^2 + L_2^2)^{1/2} .$$

As the operator contained in the square root on the right hand side of this formula is positive definite

$$P \| L_0 f \| > \| L_1 f \|$$

where f is a vector in Hilbert space in the common domain of the operators L_0 and L_1. However, if in the limit $P \to 0$ the norm of L_1, is finite non-zero, this equation shows that the norm of L_0 must diverge. Thus the operators L_0 and L_1, have no common domain of definition in Hilbert space. A similar argument also excludes a common domain for the operators L_0 and L_2. This argument is not valid however in the case that L_1 and L_2 are of zero norm, but this leads naturally to the unfaithful representations of the contracted group.

In the previous section we have shown that a negative eigenvalue of W is associated with the 'discrete class' of representations $D_l^{\pm}$ of the three dimensional homogeneous Lorentz group with the exception of the representations $D_{1/2}^{\pm}$. However these two latter representations are easily seen to be reducible components of the 'continuous' representation $C_{1/2}$ and the ensuing discussion of the contraction of the representations C_q contains a discussion of these representations as a special case.

The 'continuous class' of representations can be contracted to either faithful representations or non-faithful representations. In the form that the representation has been given in section 5 the simple limit, P converges to zero, leads to unfaithful representations. To obtain faithful representations of the contracted group we must consider the continuous sequence of representations C_q as q diverges. If the limit is chosen such that $q P \to -k^2$ when q diverges and P simultaneously goes to zero we see from (5.6) that the representations do indeed converge to the faithful representations E_k of the two dimensional Euclidean group given by (5.8).

Thus we have shown that the representations E_k may be obtained either from the representations of the rotation group or from the 'continuous' representations of the homogeneous Lorentz group, if the representations are contracted in the appropriate manner. The method of contraction of the representations is basically the same in both cases, a sequence of non-equivalent representations must be considered which converges to the appropriate contracted representation. There is however a basic difference between the two limits which is essentially connected to the fact that the transition from the compact group is necessarily a limit through a discrete sequence of finite dimensional representations to an infinite dimensional representation whilst the transition from the non-compact group is a limit through a continuous sequence of infinite dimensional representations.

To illustrate this difference we consider a group G which has a single invariant W and irreducible unitary representations D_W. If we now apply a linear homogeneous non-singular transformation $U(P)$ to the infinitesimal generators of G we obtain an isomorphic group $\tilde{G}$ with irreducible unitary representations $\tilde{D}_{\tilde{W}}(P)$ defined by

$$\tilde{D}_{\tilde{W}}(P) = U(P)\, D_W\,.$$

The spectrum of $\tilde{W}$ for which these representations are defined is dependent both on P and the spectrum of W for which the representations D_W are defined. In both the case of the three dimensional rotation group and the case of the three dimensional homogeneous Lorentz group

$$\tilde{W} = |P|\, W\,.$$

The spectrum of W for the rotation group is discrete i.e. it equals $l\,(l+1)$ where l is integer or half-integer, so that the spectrum of $\tilde{W}$, obtained by multiplying by the scale factor modulus P, is disjoint from the spectrum of W. However, for the 'continuous' class of representations of the three dimensional homogeneous Lorentz group C_q the spectra of $\tilde{W}$ and W are the same, because the spectrum of W is all positive real numbers q. Hence for these latter representations we may write

$$\tilde{C}_q(P) = U(P)\, C_{q/|P|}$$

and both sides of this equation are defined. Thus we have representations of all the isomorphic groups, corresponding to different values of P, each of which has the same value for the group invariant. It follows from the above that the representations E_k of the contracted group are given by

$$E_k = \lim_{P\to 0} \tilde{C}_k(P) = \lim_{P\to 0} U(P)\, C_{k/|P|}\,.$$

7. Conclusion

It has been shown in the foregoing sections the representations of the inhomogeneous Lorentz group may be classified by two invariants P and W. The invariant P represents the square of the physical mass M of the system represented and when this mass is real non-zero W is connected to the total spin S of the system by

$$W = M^2\, s\,(s+1)\,.$$

If the mass is zero there are representations which describe the known transformation properties of the neutrino and photon, and these representations have W equal to zero. There are also zero mass representations with W real and positive and these representations interpreted physically would describe particles with infinite angular momentum and an infinite number of helicity states.

If the representation of the infinitesimal elements of the inhomogeneous Lorentz group are now written as $\mathfrak{D}(M, W)$ it follows that the representations $\mathfrak{D}(0, k)$ may be obtained as limits of the non-zero mass representations in two ways:

either 1) $$\lim_{\epsilon \to 0} \mathfrak{D}(i\,\epsilon, k) = \mathfrak{D}(0, k)$$

or 2) $$\lim_{\epsilon \to 0} \lim_{s \to \infty} \mathfrak{D}\big(\epsilon, \epsilon^2 s\,(s+1)\big) = \mathfrak{D}(0, k)$$

where the limit on S is stepwise such that

$$\lim_{\epsilon \to 0} \lim_{s \to \infty} \epsilon^2 s^2 = k\,.$$

Therefore, although it is not possible to dissociate the 'strange' representations of zero-mass from a limited form of the representations for real mass, it does seem more reasonable to associate these representations with the limit of representations describing particles with imaginary mass.

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