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TORSION AUTOMORPHISMS OF SIMPLE LIE ALGEBRAS

by Mark REEDER *)

1. INTRODUCTION

An automorphism σ of a simple finite-dimensional complex Lie algebra $\mathfrak{g}$ is called *torsion*, if σ has finite order in the group $\mathrm{Aut}(\mathfrak{g})$ of all automorphisms of $\mathfrak{g}$. The torsion automorphisms of $\mathfrak{g}$ were classified by Victor Kac in [12], as an application of his results on infinite-dimensional Lie algebras.

Those torsion automorphisms contained in the identity component $G = \mathrm{Aut}(\mathfrak{g})^\circ$ are called *inner*; they were classified in 1927 by Élie Cartan [6] who used (and perhaps introduced) the affine Weyl group and the geometry of alcoves for this purpose. This paper extends Cartan's method to cover all torsion automorphisms of $\mathfrak{g}$, thereby recovering Kac's classification directly from the geometry of the affine Weyl group, without the use of infinite-dimensional Lie algebras.

Kac's classification can be roughly stated as follows. Each symmetry ϑ of the Dynkin graph $\mathcal{D}(\mathfrak{g})$ of $\mathfrak{g}$ extends to a certain kind of automorphism of $\mathfrak{g}$, which we again denote by ϑ, called a *pinned automorphism*. The pinned automorphisms represent the cosets of G in $\mathrm{Aut}(\mathfrak{g})$, and the order of any torsion element in $G\vartheta$ is divisible by the order f of ϑ. For a given pinned automorphism ϑ of $\mathfrak{g}$, Kac defines a certain vector $(b_0, b_1, \dots, b_k)$ of positive integers. Here k is the number of ϑ-orbits on the nodes of $\mathcal{D}(\mathfrak{g})$. Then the G-conjugacy classes of elements in $G\vartheta$ of order m are parametrized by *Kac coordinates*. These are vectors $(s_0, s_1, \dots, s_k)$ of nonnegative relatively prime integers s_i satisfying

$$f \cdot \sum_{i=0}^{k} b_i s_i = m \, .$$

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For a more precise statement see Theorem 3.7. When $\vartheta = 1$, the integers b_i are the coefficients of the highest root in $\mathfrak{g}$. For nontrivial ϑ, the b_i's are closely related to the coefficients of the highest short root in the fixed-point subalgebra $\mathfrak{g}^{\vartheta}$.

The desire to understand torsion automorphisms and their Kac coordinates in simple terms arose from the work of Benedict Gross and myself on adjoint gamma factors of discrete Langlands parameters attached to representations of p-adic groups [9]. Jean-Pierre Serre pointed us to Cartan's paper, which led to the approach to Kac's classification presented here. A brief sketch of an approach similar to this is given in [15]. However, the examples and details worked out herein were useful to us, and I hope they will be useful to others.

Throughout, I make frequent use of Kostant's theory of the principal PGL_2 and of conjugacy results due to Steinberg. I give many examples of interesting torsion classes and show how to compute their Kac coordinates. For the classical Lie algebras, the torsion automorphisms can be classified using linear algebra; see Section 4, where each simple Lie algebra is examined separately. I include some facts about centers and component groups of centralizers that may not have appeared in the literature, and the last section gives a twisted analogue (Proposition 5.1) of a result of Kostant on principal elements. These complements are used in [9].

Since [9] was written, Gross, Jiu-Kang Yu and I have found further connections between torsion automorphisms of simple Lie algebras and the representation theory of p-adic groups. These applications will not be explained here, but they have informed some of the examples below.

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2. Inner automorphisms

Reviewing Cartan's classification [6] of inner automorphisms will serve to introduce some of the structure in what follows, and as a template for the general case. See [18] for an introduction, and [3] for foundations of the theory of root systems as used below.

2.1 BASIC STRUCTURE

Let $\mathrm{Aut}(\mathfrak{g})$ be the group of automorphisms of a simple complex Lie algebra $\mathfrak{g}$. The identity component $G = \mathrm{Aut}(\mathfrak{g})^\circ$ is a simple complex algebraic group with trivial center and Lie algebra $\mathfrak{g}$. Let $T \subset B$ be a maximal torus and a Borel subgroup of G and let Φ be the set of roots of T in G, with positive system Φ^+ given by the roots in B and let $\Delta = \{\alpha_1, \dots, \alpha_\ell\} \subset \Phi^+$ be the corresponding simple roots, where ℓ is the rank of G.

Since G has trivial center, Δ is a $\mathbf{Z}$-basis of the *weight lattice* $X = X^*(T)$ of algebraic homomorphisms $T \to \mathbf{C}^\times$. We let

$$\langle\ ,\ \rangle \colon X \times Y \longrightarrow \mathbf{Z}$$

be the natural pairing between $X = X^*(T)$ and the *co-weight lattice* $Y = X_*(T)$ of algebraic homomorphisms $\mathbf{C}^\times \to T$. Let $\{\check{\omega}_1, \dots, \check{\omega}_\ell\}$ be the $\mathbf{Z}$-basis of Y consisting of fundamental co-weights dual to Δ:

$$\langle \alpha_i, \check{\omega}_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

The *Weyl group* $W = N/T$, where N is the normalizer of T in G, acts on the real vector space

$$V = \mathbf{R} \otimes Y$$

as a group generated by reflections $r_1, \dots r_\ell$, where r_i fixes the hyperplane $(\alpha_i = 0)$ pointwise. We regard V as the Lie algebra of the maximal compact subtorus $S \subset T$, via the exponential map

$$\exp \colon V \to S,$$

which is a surjective group homomorphism defined by the property:

$$\alpha(\exp(x)) = e^{2\pi i \langle \alpha, x \rangle} \quad \text{for all } \alpha \in \Phi,$$

where on the left side we view α as a character of T restricted to S and on the right side α is a linear functional on V. Then

$$Y = \ker \exp = \{x \in V \,:\, \langle \alpha, x \rangle \in \mathbf{Z} \quad \forall \alpha \in \Phi\},$$

so $\exp$ induces an isomorphism

$$V/Y \xrightarrow{\sim} S.$$

2.2 TORSION ELEMENTS IN G

An element $s \in G$ is *semisimple* if s acts diagonalizably on $\mathfrak{g}$. Any semisimple element is G-conjugate to an element of T, and two elements of T are G-conjugate if and only if they are W-conjugate. Thus, the set of semisimple conjugacy classes in G is in bijection with the set of W-orbits on T.

Any torsion element $s \in G$ is semisimple and is G-conjugate to an element of S; we have $s = \exp(x)$, for some $x \in V_{\mathbf{Q}} := \mathbf{Q} \otimes Y$. Our discussion so far shows that two elements $x, x' \in V_{\mathbf{Q}}$ give G-conjugate elements $\exp(x)$ and $\exp(x')$ if and only if x, x' are conjugate under the *extended affine Weyl group*

$$\widetilde{W} := W \ltimes Y,$$

where Y acts on V by translations. This analysis by Cartan in [6, Part I] is perhaps the first appearance of the extended affine Weyl group in the literature.

The (unextended) *affine Weyl group* is a normal subgroup $\widetilde{W}^{\circ} \lhd \widetilde{W}$ which can be described in two ways: First,

$$\widetilde{W}^{\circ} = W \ltimes \mathbf{Z}\check{\Phi},$$

where $\mathbf{Z}\check{\Phi} \subset Y$ is the lattice of co-roots of T in G. The group $\widetilde{W}^{\circ}$ is also the group of affine transformations of V generated by the reflections in V about the affine *root hyperplanes* with equations $\alpha = n$, where $\alpha \in \Phi$ and $n \in \mathbf{Z}$. In fact, $\widetilde{W}^{\circ}$ is generated by $\ell + 1$ such affine reflections chosen as follows. An *alcove* is a connected component C of the set of points in V not lying on any root hyperplane. A *wall* of C is a root hyperplane H meeting the closure of C in an open subset of H. Each alcove has $\ell + 1$ walls. The two key facts [3, V.3.2] are first, that $\widetilde{W}^{\circ}$ is a Coxeter group generated by the $\ell + 1$ reflections about the walls of any fixed alcove, and second, that $\widetilde{W}^{\circ}$ permutes the alcoves in V freely and transitively.

The basis Δ determines a particular alcove, as follows. Let $\widetilde{\alpha}_0 = \sum_{i=1}^{\ell} a_i\,\alpha_i$ be the highest root with respect to Δ (here the a_i are positive integers), let α_0 be the affine linear function $1 - \widetilde{\alpha}_0$ on V and set $a_0 = 1$, so that

$$\sum_{i=0}^{\ell} a_i\,\alpha_i \equiv 1\,.$$

Then the *alcove determined by* Δ is the intersection of half-spaces:

$$C = \{x \in V : \langle \alpha_i, x \rangle > 0 \quad \text{for} \quad 0 \le i \le \ell\}.$$

It is convenient to set

$$\check{\omega}_0 = 0 \in V.$$

Then we can write the closure $\overline{C}$ of C in barycentric coordinates as

$$\overline{C} = \Big\{ \sum_{i=0}^{\ell} x_i \check{\omega}_i \ : \ x_i \geq 0 \quad \text{and} \quad \sum_{i=0}^{\ell} a_i x_i = 1 \Big\}.$$

Thus, $\overline{C}$ is the convex hull of its *vertices*

$$v_i := a_i^{-1} \check{\omega}_i \,, \quad \text{for} \quad 0 \leq i \leq \ell \,.$$

Note that $v_0 = \check{\omega}_0 = 0$ is one of the vertices of $\overline{C}$.

Since the affine Weyl group $\widetilde{W}^{\circ}$ is transitive on alcoves, so is the extended affine Weyl group $\widetilde{W}$. Hence the closure $\overline{C}$ meets all $\widetilde{W}$-orbits in V. This means that each torsion element $s \in G$ is conjugate to $\exp(x)$ for some $x \in \overline{C} \cap V_{\mathbf{Q}}$. Unlike $\widetilde{W}^{\circ}$, however, $\widetilde{W}$ does not act freely on the alcoves in V, so we must also take into account the alcove stabilizer

$$\Omega := \{ \rho \in \widetilde{W} \ : \ \rho \cdot C = C \} \,,$$

which is a complement to $\widetilde{W}^{\circ}$ in $\widetilde{W}$:

$$\widetilde{W} = \Omega \ltimes \widetilde{W}^{\circ}. \tag{2.1}$$

If x and x' are in $\overline{C}$, the elements $\exp(x)$ and $\exp(x')$ are G-conjugate if and only if x and x' are conjugate under Ω. Pictures of C in the case $\ell = 2$, along with fundamental domains for Ω in C, can be found in [6, p. 224]. See also Section 2.5 below.

Let $x \in \overline{C}$ and suppose $\exp(x)$ is a torsion element of order m. Since $\exp(mx) = 1$, there are nonnegative integers $s_1, \ldots, s_\ell$ such that

$$x = \frac{1}{m} \sum_{i=1}^{\ell} s_i \, \check{\omega}_i \,. \tag{2.2}$$

Since $\exp(x)$ has exact order m, it follows that $\gcd\{m, s_1, \ldots, s_\ell\} = 1$. As $x \in \overline{C}$, we have

$$0 \leq \langle \alpha_0, x \rangle = 1 - \frac{1}{m} \sum_{i=0}^{\ell} a_i s_i \,,$$

so that the integer $s_0 := m - \sum_{i=1}^{\ell} a_i s_i$ is ≥ 0 and $s_0, s_1, \ldots, s_\ell$ satisfy the equation

$$\sum_{i=0}^{\ell} a_i s_i = m \,, \tag{2.3}$$

where $a_0 = 1$, and $\gcd\{s_0, \ldots, s_\ell\} = 1$. We call the sequence $(s_0, s_1, \ldots, s_\ell)$ the *Kac coordinates* of s. They determine the action of s on $\mathfrak{g}$ explicitly as follows. If $\alpha = \sum_{i=1}^{\ell} c_i \alpha_i \in \Phi$ and we set $c \cdot s = \sum_{i=1}^{\ell} c_i s_i$, then s acts on the root space $\mathfrak{g}_\alpha$ by the scalar $\zeta^{c\cdot s}$, where $\zeta = \exp(2\pi\sqrt{-1}/m)$. Two such elements $s = \exp(x)$ and $s' = \exp(x')$ are G-conjugate if and only if their Kac coordinates $(s_0, \ldots, s_\ell)$ and $(s'_0, \ldots, s'_\ell)$ are conjugate under the permutation action of Ω on $\{0, 1, \ldots, \ell\}$ induced by its action on the vertices of C. To visualize this action, it is convenient to regard $(s_0, \ldots, s_\ell)$ as a labelling of the nodes of the extended Dynkin diagram $\widetilde{\mathcal{D}}(\mathfrak{g})$. These nodes correspond to the vertices of C and Ω acts on the labellings via symmetries of $\widetilde{\mathcal{D}}(\mathfrak{g})$. Thus, s and s' are G-conjugate if and only if their labellings of $\widetilde{\mathcal{D}}(\mathfrak{g})$ are conjugate under Ω.

To describe the group Ω, we first observe from (2.1) and the definitions of $\widetilde{W}$, $\widetilde{W}^\circ$ that $\Omega \simeq Y/\mathbf{Z}\check{\Phi}$. In fact, each coset in $Y/\mathbf{Z}\check{\Phi}$ contains a unique co-weight $\check{\omega}_i$ which is a vertex of C; that is, we have $a_i = 1$ and $v_i = \check{\omega}_i$. Such co-weights are called *minuscule*. For each minuscule co-weight $\check{\omega}_i$ there is a unique element $\rho_i \in \Omega$ such that $\rho_i \cdot v_0 = v_i$. This correspondence is an explicit group isomorphism $Y/\mathbf{Z}\check{\Phi} \simeq \Omega$, and we have

$$\Omega = \{\rho_i \,:\, a_i = 1\}.$$

The group Ω also has a topological interpretation: The lattice $\mathbf{Z}\check{\Phi}$ is the co-weight lattice $X_*(T')$, where T' is a maximal torus in the simply-connected cover G' of G. It follows that $\Omega \simeq Y/\mathbf{Z}\check{\Phi}$ is isomorphic to the fundamental group $\pi_1(G)$ of G. For more details on the group Ω, see [3, VI.2.3].

The above discussion is essentially the classification of torsion inner automorphisms given by Cartan in [6, Part I.4-6]. The minuscule vertices appear in [6, Part I.7], where they are denoted by $O_1, \ldots, O_{h-1}$, and are used by Cartan to study $\pi_1(G)$.

EXAMPLE 1. Let $\mathfrak{g} = \mathfrak{sl}_{\ell+1}$, so that $G = PGL_{\ell+1}$ is the quotient of $GL_{\ell+1}$ by its center, which consists of scalar matrices. Let $[t_1, \ldots, t_{\ell+1}]$ be the image in G of a diagonal matrix $\mathrm{diag}(t_1, t_2, \ldots, t_{\ell+1}) \in GL_{\ell+1}$. All the coefficients $a_i = 1$, so that an element in G of order m has Kac coordinates $(s_0, \ldots, s_\ell)$, where the relatively prime non-negative integers s_i satisfy $s_0 + s_1 + \cdots + s_\ell = m$. An element $s \in G$ with these Kac coordinates is given by

$$s = [\zeta^{s_1+s_2+\cdots+s_\ell}, \zeta^{s_2+s_3+\cdots+s_\ell}, \ldots, \zeta^{s_\ell}, 1], \tag{2.4}$$

where $\zeta = \exp(2\pi\sqrt{-1}/m)$. One can see this from equation (2.2) as follows. We have

$$s = \exp(x) = \prod_{i=1}^{\ell} \exp\big(\frac{s_i}{m}\check{\omega}_i\big).$$

The vector space V is the quotient of $\mathbf{R}^{\ell+1}$ by the diagonal, and the fundamental co-weight $\check{\omega}_i$ is the image in V of the vector $(1,\dots,1,0,\dots,0)$, with i entries equal to 1. On the line through $\check{\omega}_i$ the exponential map is given by

$$\exp(t\check{\omega}_i) = [e^{2\pi t\sqrt{-1}},\dots,e^{2\pi t\sqrt{-1}},1,\dots,1],$$

so that

$$\exp\bigl(\tfrac{s_i}{m}\check{\omega}_i\bigr) = [\zeta^{s_i},\dots,\zeta^{s_i},1,\dots,1],$$

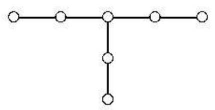
where ζ^{s_i} appears i times; taking the product over i gives (2.4).

It may seem that s_0 only appears in (2.4) indirectly, via the fact that ζ has order $m = s_0+\cdots+s_\ell$. In fact s_0 appears on equal footing with the other s_i's. To see this, one can check that

$$(2.5)\quad [\zeta^{s_2+s_3+\cdots+s_\ell+s_0},\zeta^{s_3+s_4+\cdots+s_\ell+s_0},\dots,\zeta^{s_0},1] = [\zeta^{s_2+s_3+\cdots+s_\ell},\zeta^{s_3+s_4+\cdots+s_\ell},\dots,1,\zeta^{s_1+s_2+\cdots+s_\ell}],$$

which is conjugate to the element s in (2.4). This reflects the fact that $\widetilde{\mathcal{D}}(\mathfrak{g})$ is an $(\ell+1)$-gon, on which the group $\Omega\simeq\mathbf{Z}/(\ell+1)$ acts by rotations.

EXAMPLE 2. Fortunately, it is not necessary to have explicit realizations of co-weights or group elements to get concrete information about torsion classes in G. We illustrate this by finding the classes of order three in $G=E_6$. The diagram $\widetilde{\mathcal{D}}(\mathfrak{e}_6)$ and the coefficients a_i are given by



$$\begin{array}{ccccc} 1 & 2 & 3 & 2 & 1 \\ & & 2 & & \\ & & 1 & & \end{array}$$

and the group Ω is cyclic of order three, acting on $\widetilde{\mathcal{D}}(\mathfrak{e}_6)$ by rotations. There are five classes of elements of order three, with Kac coordinates

$$\begin{array}{ccccc} 0\;1\;0\;0\;0 & 0\;0\;0\;1\;0 & 0\;0\;0\;0\;0 & 0\;0\;1\;0\;0 & 1\;0\;0\;0\;1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{array}$$

The first two labellings are conjugate by the reflection of the diagram about the vertical axis. This means that the union of these distinct classes in G forms a single class in $\mathrm{Aut}(\mathfrak{e}_6)$, which contains G with index two.

2.3 COMPUTING KAC COORDINATES

In practice, one often seeks the Kac coordinates of a semisimple element $s=\exp(x)$ of known order, for which x lies in V but not in $\overline{C}$. For this we have the following algorithm.

Let $s = \exp(x)$ have order m, where $x \in V$ is arbitrary. Again there are integers $s_1, \dots, s_\ell$ with $\gcd\{m, s_1, \dots, s_\ell\} = 1$ such that

$$x = \frac{1}{m}\sum_{i=1}^{\ell} s_i\,\check{\omega}_i .$$

As before, we also define s_0 by the equation $\sum_{i=0}^{\ell} a_i s_i = m$, and we have

$$\langle \alpha_i, x\rangle = \frac{s_i}{m} \quad \text{for } i = 0, 1, \dots, \ell .$$

The difference now is that if $x \notin \overline{C}$, then some of the s_i's will be negative.

The algorithm runs as follows. If all $s_j \geq 0$, then x is already in $\overline{C}$. Otherwise, select some $s_j < 0$ and replace $(s_0, \dots, s_\ell)$ by $(s_0', \dots, s_\ell')$, where

$$s_i' = s_i - \langle \alpha_i, \check{\alpha}_j\rangle s_j . \tag{2.6}$$

Repeat the previous steps with the new coordinates $(s_0', \dots, s_\ell')$. Eventually one arrives at coordinates s_i' which are all ≥ 0, and these are the Kac coordinates of s.

To see that the algorithm works, we recall that the affine Weyl group $\widetilde{W}^\circ$ is a Coxeter group generated by the reflections $r_0, \dots, r_\ell$ about the walls of C; these are given explicitly by the formulas

$$r_j \cdot x = x - \langle \alpha_j, x\rangle \check{\alpha}_j ,$$

where $\check{\alpha}_j$ is the co-root corresponding to the gradient of the affine root α_j. For $w \in \widetilde{W}^\circ$, let $\ell(w)$ be the minimal length of a word expressing w in the generators $\{r_i\}$. We have $\ell(r_j w) < \ell(w)$ if and only if $w^{-1}\alpha_j$ is negative on C. For any $x \in V$, set

$$d(x) = \ell(w) ,$$

where $w \in \widetilde{W}^\circ$ is of minimal length such that the point $y := w^{-1}x$ is contained in $\overline{C}$. Clearly $d(x) = 0$ if and only if $x \in \overline{C}$. Now the transformed coordinates $(s_0', \dots, s_\ell')$ given by (2.6) are those of $r_j \cdot x$. If $s_j < 0$ then

$$0 > \frac{s_j}{m} = \langle \alpha_j, x\rangle = \langle \alpha_j, w \cdot y\rangle = \langle w^{-1}\alpha_j, y\rangle ,$$

so that $w^{-1}\alpha_j < 0$ on C. Hence $\ell(r_j w) < \ell(w)$. Since $y = (r_j w)^{-1} r_j \cdot x$, we have

$$d(r_j x) \leq \ell(r_j w) < \ell(w) = d(x) .$$

Hence the algorithm succeeds in $\ell(w)$ steps.

Regard the vector $(s_0, \dots, s_\ell)$ as a labelling of the affine Dynkin diagram $\widetilde{\mathcal{D}}(\mathfrak{g})$ by placing s_i on the i^{th} node of $\widetilde{\mathcal{D}}(\mathfrak{g})$. We can implement the algorithm by manipulating the labelling, as illustrated in the following example. Let $G = G_2$, where the coefficients a_i are given by $\overset{1}{\circ}\!-\!\!\!-\!\overset{2}{\circ}\!\Rrightarrow\!\overset{3}{\circ}$. Consider the element $t \in G$ of order five with Kac diagram $\overset{0}{\circ}\!-\!\!\!-\!\overset{1}{\circ}\!\Rrightarrow\!\overset{1}{\circ}$. Let us compute the conjugacy class of $s = t^2$, which also has order five. We have $t = \exp(y)$, where $y = \frac{1}{5}(\check{\omega}_1 + \check{\omega}_2) \in \overline{C}$, and $s = \exp(x)$, where $x = 2y = \frac{1}{5}(2\check{\omega}_1 + 2\check{\omega}_2) \notin \overline{C}$. Thus, $s_1 = s_2 = 2$ and $s_0 = 5 - (2\cdot 2 + 3\cdot 2) = -5$, so the algorithm runs as

$$\overset{-5}{\circ}\!-\!\!\!-\!\overset{2}{\circ}\!\Rrightarrow\!\overset{2}{\circ} \xrightarrow{r_0} \overset{5}{\circ}\!-\!\!\!-\!\overset{-3}{\circ}\!\Rrightarrow\!\overset{2}{\circ} \xrightarrow{r_1} \overset{2}{\circ}\!-\!\!\!-\!\overset{3}{\circ}\!\Rrightarrow\!\overset{-1}{\circ} \xrightarrow{r_2} \overset{2}{\circ}\!-\!\!\!-\!\overset{0}{\circ}\!\Rrightarrow\!\overset{1}{\circ}.$$

The final diagram gives the Kac coordinates of t^2, and shows that t^2 is not conjugate to t in G_2.

2.4 Centralizers

The centralizer $C_G(s)$ of a torsion element $s \in G$ can be described in terms of the geometry of the alcove C and the action on C by Ω. The closure $\overline{C}$ is partitioned into a disjoint union of $2^{\ell+1} - 1$ facets:

$$\overline{C} = \bigcup_J C^J,$$

indexed by the proper subsets $J \subsetneq \{0, \dots, \ell\}$. The facet C^J consists of the points $x \in \overline{C}$ such that $\langle \alpha_i, x\rangle = 0$ for $i \in J$ and $\langle \alpha_i, x\rangle > 0$ for $i \notin J$. For example, $C^{\varnothing} = C$ and for $J = \{0, \dots, \ell\} - \{i\}$ we have $C^J = \{v_i\}$. Let Φ_J be the set of roots in Φ which are constant on C^J. Then Φ_J is a root subsystem of Φ of rank $|J|$, with basis $\Delta_J := \{\alpha_j : j \in J\}$. If $x \in C^J \cap V_{\mathbf{Q}}$, the Kac coordinates $(s_0, \dots, s_\ell)$ of the torsion automorphism $s = \exp(x)$ have $s_j = 0$ if and only if $j \in J$.

The subalgebra $\mathfrak{g}^s$ of vectors in $\mathfrak{g}$ fixed by s is reductive, and depends only on J. Namely,

$$\mathfrak{g}^s = \mathfrak{t} \oplus \sum_{\alpha \in \Phi_J} \mathfrak{g}_\alpha . \tag{2.7}$$

The (unextended) Dynkin diagram $\mathcal{D}(\mathfrak{g}^s)$ of $\mathfrak{g}^s$ is the subgraph of $\widetilde{\mathcal{D}}(\mathfrak{g})$ supported on J.

For example, the element s is *regular* if $\mathfrak{g}^s = \mathfrak{t}$. This occurs exactly when $x \in C$, or equivalently, when all $s_i > 0$. Taking all $s_i = 1$ gives the unique class of regular elements of minimal order

$$h := a_0 + a_1 + \cdots + a_\ell , \tag{2.8}$$

the *Coxeter number* of G [3, VI.1.11]. We will return to this in Section 2.5.

The *identity component* $C_G(s)^\circ$ of $C_G(s)$ is determined by $\mathfrak{g}^s$, hence it too depends only on J. Explicitly, the root datum of $C_G(s)^\circ$ is that of G but with the roots Φ and co-roots $\check{\Phi}$ replaced by the roots Φ_J and the co-roots $\check{\Phi}_J = \{\check{\alpha} : \alpha \in \Phi_J\}$, respectively.

However, the component group A_s of $C_G(s)$ is more sensitive: it depends on the actual point $x \in C^J$. For example, if $\mathfrak{g} = \mathfrak{sl}_n$ then V is the set of vectors in $\mathbf{R}^n$ whose coordinates sum to zero. The simple roots $\alpha_i = x_i - x_{i+1}$ define the alcove

$$C = \{(x_1, \dots, x_n) \in V : x_n + 1 > x_1 > x_2 > \cdots > x_n\}.$$

There is an open dense subset $U \subset C$ for which $C_G(s) = T$ when $s \in \exp(U)$. On the other hand, at the barycenter $\bar{x} := \frac{1}{2n}(n-1, n-3, \dots, 3-n, 1-n)$ of C, the element $\bar{s} = \exp \bar{x}$ has order n and has Kac coordinates $(1, 1, \dots, 1)$. The centralizer $C_G(\bar{s})$ of $\bar{s}$ in $G = PGL_n(\mathbf{C})$ is a semidirect product $T \rtimes \langle\sigma\rangle$, where $\sigma \in N$ is a lift of a Coxeter element $w \in W$ and has order n. Since it is the barycenter of C, the point $\bar{x}$ is fixed by the group Ω which is also cyclic of order n, generated by the affine transformation $\rho_1 : (x_1, x_2, \dots, x_n) \mapsto \left(x_n + 1 - \frac{1}{n},\ x_1 - \frac{1}{n},\ x_2 - \frac{1}{n}, \dots, x_{n-1} - \frac{1}{n}\right)$ and we can take w to be the projection of ρ_1 to W. This example is generalized in the next section.

The relation between A_s and the geometry of C is governed by the alcove stabilizer Ω, as follows.

PROPOSITION 2.1. *For $s = \exp(x)$ with $x \in \overline{C}$, the component group A_s of $C_G(s)$ is isomorphic to the stabilizer $\Omega_x = \{\rho \in \Omega : \rho \cdot x = x\}$.*

Proof. Let $\widetilde{W}_x = \{w \in \widetilde{W} : w \cdot x = x\}$ be the stabilizer of x in $\widetilde{W}$. This group is finite, and its normal subgroup $\widetilde{W}_x^\circ$, generated by reflections about hyperplanes through x, acts simply-transitively on the set of alcoves containing x in their closure [3, V, Thms 1, 2]. It follows that $\widetilde{W}_x$ decomposes as

$$\widetilde{W}_x = \Omega_x \ltimes \widetilde{W}_x^\circ. \tag{2.9}$$

On the other hand, let W_s be the stabilizer of s in W. The projection $\pi : \widetilde{W} \to W$ sends $\widetilde{W}_x$ to W_s. Since $\widetilde{W}_x$ is finite and Y is torsion-free, the map π is injective on $\widetilde{W}_x$. If $w \cdot s = s$, then $w \cdot x \in x + Y$, so there is $y \in Y$ such that $t_y w \cdot x = x$. It follows that π restricts to an isomorphism $\widetilde{W}_x \xrightarrow{\sim} W_s$.

The image $\pi(\widetilde{W}_x^\circ)$ is the subgroup $W_s^\circ \subset W_s$ generated by reflections for the roots in $\Phi_s^+ := \{\alpha \in \Phi^+ : \alpha(s) = 1\}$. Hence π induces an isomorphism

$$\Omega_x \xrightarrow{\sim} W_s/W_s^\circ . \tag{2.10}$$

The group $C_G(s)^\circ$ is reductive, with maximal torus T and Borel subgroup $B_s = B \cap C_G(s)$. Put

$$N_s = N \cap C_G(s) \quad \text{and} \quad N_s^\circ = N \cap C_G(s)^\circ .$$

Then $W_s^\circ = N_s^\circ/T$ is the Weyl group of T in $C_G(s)^\circ$ [7, 3.5] and

$$W_s/W_s^\circ \simeq N_s/N_s^\circ . \tag{2.11}$$

Since $C_G(s)^\circ$ acts transitively on its Borel subgroups and N_s° acts transitively on the Borel subgroups of $C_G(s)^\circ$ containing T, it follows that the inclusion $N_s \hookrightarrow C_G(s)$ gives an isomorphism

$$N_s/N_s^\circ \simeq A_s . \tag{2.12}$$

Combining equations (2.10), (2.11) and (2.12), we get $\Omega_x \simeq A_s$, as claimed. Finally, since we have seen that Ω is abelian, it follows that A_s is abelian.

In the example for $G = E_6$ in Section 2.2, the first three classes have trivial stabilizer in Ω, hence have connected centralizer in G, while the centralizers of

$$\begin{matrix} 0 & 0 & 1 & 0 & 0 \\ & & 0 & & \\ & & 0 & & \end{matrix} \qquad \text{and} \qquad \begin{matrix} 1 & 0 & 0 & 0 & 1 \\ & & 0 & & \\ & & 1 & & \end{matrix}$$

have three components.

REMARKS. 1) The projection of Ω into W is the subgroup Γ of W preserving the set $\widetilde{\Delta} = \{-\widetilde{\alpha}_0, \alpha_1, \ldots, \alpha_\ell\}$ of simple roots augmented by the lowest root, and Ω_x projects isomorphically onto the subgroup Γ_s of Γ preserving the base $\widetilde{\Delta} \cap \Phi_s$ of Φ_s. This group Γ_s is a complement to W_s° in W_s.

2) Recall that we can identify Ω with the fundamental group $\pi_1(G)$ of G. From this point of view, the embedding $A_s \hookrightarrow \pi_1(G)$ can be seen as follows. Let $G' \to G$ be the simply-connected covering of G, with kernel $\pi_1(G)$. Choose a lift $g' \in G'$ of every element $g \in C_G(s)$. Then the commutator $g \mapsto [g', s']$ induces a well-defined homomorphism $A_s \to \pi_1(G)$ which is injective, since the centralizer of s' in G' is connected [21, 8.1].

2.5 Kac coordinates of principal elements

The smallest simple Lie algebra is $\mathfrak{sl}_2$, consisting of all 2×2 matrices of trace zero, with bracket $[A, B] = AB - BA$. In [14], Kostant showed that $\mathfrak{sl}_2$ plays a powerful role in the structure theory of an arbitrary simple complex Lie algebra $\mathfrak{g}$. There are finitely many embeddings of $\mathfrak{sl}_2$ in $\mathfrak{g}$, up to conjugacy by $G = \mathrm{Aut}(\mathfrak{g})^\circ$. One class of embeddings is distinguished by its behavior on Cartan subalgebras. Fix a Cartan subalgebra $\mathfrak{t}_0$ of $\mathfrak{sl}_2$. For example, we could take $\mathfrak{t}_0$ to be the diagonal matrices in $\mathfrak{sl}_2$. Each embedding $\mathfrak{sl}_2 \hookrightarrow \mathfrak{g}$ sends $\mathfrak{t}_0$ into a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$, and usually into infinitely many such $\mathfrak{t}$'s. However, there is exactly one G-conjugacy class of embeddings $\varphi\colon \mathfrak{sl}_2 \hookrightarrow \mathfrak{g}$ with the property that $\varphi(\mathfrak{t}_0)$ is contained in a *unique* Cartan subalgebra of $\mathfrak{g}$.

This has the following implication on the level of groups. The automorphism group of $\mathfrak{sl}_2$ is PGL_2, the quotient of GL_2 by the scalar matrices, which acts on $\mathfrak{sl}_2$ by conjugation. The above facts mean that there is a unique conjugacy class of algebraic subgroups $G_0 \subset G$ which are isomorphic to PGL_2, with the property that any maximal torus T_0 of G_0 is contained in a unique maximal torus T of G. Such a subgroup G_0 is called a *principal* PGL_2 in G. We say that an element $s \in G$ is a *principal element* if s lies in some principal PGL_2 in G. In this section we study the Kac coordinates of principal torsion elements of G.

We can choose G_0, a principal PGL_2 in G, along with a maximal torus T_0 in G_0, so that T is the unique maximal torus of G containing T_0 and the simple roots $\alpha_1, \ldots, \alpha_\ell$ of T each restrict to the same root α of T_0 in G_0. This means that T_0 is the closed subgroup of T defined by the equations $\alpha_1 = \alpha_2 = \cdots = \alpha_\ell$, and $X_*(T_0)$ is the subgroup of $X_*(T)$ generated by the co-weight $\check{\rho} \in X_*(T)$ defined by the conditions $\langle \alpha_i, \check{\rho} \rangle = 1$ for $1 \le i \le \ell$. In the line $V_0 := \mathbf{R} \otimes X_*(T_0)$ we have the alcove

$$C_0 = \{ r\check{\rho} : 0 < r < 1 \} \subset V_0 .$$

However, only part of C_0 is contained in C. Indeed, we have

$$\langle \alpha_0, r\check{\rho} \rangle = 1 - r \sum_{i=1}^{\ell} a_i = 1 - r(h-1),$$

where we recall from (2.8) that h is the Coxeter number of G. It follows that $r\check{\rho} \in \overline{C}$ if and only if $r \le (h-1)^{-1}$.

Suppose $s = \exp(r\check{\rho}) \in G_0$ has finite order $m > 1$. Then $r = n/m$ for relatively prime positive integers $n < m$. For $1 \leq i \leq \ell$ we then have

$$\alpha_i(s) = \alpha(s) = \exp(2\pi r\sqrt{-1}).$$

If $r \leq (h-1)^{-1}$, so that $r\check{\rho} \in \overline{C}$, the Kac coordinates of s are obtained as follows.

Since

$$r\check{\rho} = \frac{1}{m}\sum_{i=1}^{\ell} n\check{\omega}_i,$$

we have $s_1 = s_2 = \cdots = s_\ell = n$. Then

$$m = s_0 + \sum_{i=1}^{\ell} n \cdot a_i = s_0 + n(h-1).$$

Hence the Kac coordinates of $s = \exp(r\check{\rho})$ are

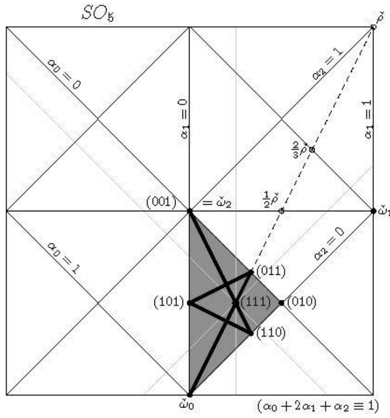
$$(n - nh + m, n, n, \ldots, n), \quad \text{when} \quad r = \frac{n}{m} \leq \frac{1}{h-1}. \tag{2.13}$$

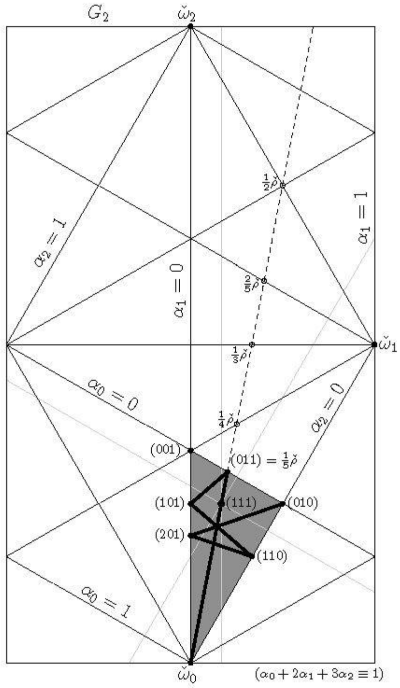
We have $x \in C$ if and only if $r < 1/(h-1)$. For this inequality to hold, we must then have $m \geq h$.

If $m = h$ then $n = 1$ and s is *Kostant's principal element*, with Kac coordinates $(1, 1, \ldots, 1)$, having the smallest possible order h of a regular torsion element in G [14]. We have $\bar{s} = \exp(\bar{x})$, where $\bar{x} = \check{\rho}/h$ is the unique point in the alcove C at which all simple affine roots take the same value, namely $1/h$ (cf. [14, 8.6]). Kostant's principal elements appeared in Section 2.4 for $G = PGL_n$. For a twisted analogue of them, see Section 5 below.

If we continue on the path $r\check{\rho}$ for $r > 1/(h-1)$, the Kac coordinates become less obvious than those of (2.13); one must use the algorithm of Section 2.3, for which the number of steps depends on r, to conjugate back into $\overline{C}$. We need only go up to $r = 1/2$, since every torsion element of G_0 is conjugate to some $\exp(r\check{\rho})$ for rational $r \in [0, 1/2]$. As we exit C at $r = 1/(h-1)$ and proceed, we enter new alcoves, creating segments in each alcove. Each of these segments is $\widetilde{W}^{\circ}$-conjugate to a unique segment in $\overline{C}$. The resulting collection of segments in $\overline{C}$ forms the path of a billiard ball with initial direction from 0 to $\check{\rho}$.

Pictures for SO_5 and G_2 are shown below. The dotted line consists of points in C_0 lying outside C and the points where the billiard path bounces off a wall of $\overline{C}$ are labelled by their Kac coordinates. The faint lines intersecting at Kostant's principal element (111) are those with equations $\alpha_i = 1/h$, for $i = 0, 1, 2$. In the picture for SO_5 we have continued out to $\check{\rho}$, to show how the symmetry $r\check{\rho} \leftrightarrow (1-r)\check{\rho}$ by conjugation in G_0 is transformed into the nontrivial symmetry of the alcove C.





2.6 Kac coordinates and regular elements in the Weyl group

In the examples above, many of the interesting torsion classes have all of their Kac coordinates $s_i \in \{0, 1\}$. Such classes often come from the Weyl group of G. For example, Kostant showed that his class of principal elements, with all $s_i = 1$, meets the normalizer N of T in a single N-conjugacy class which projects to the class of Coxeter elements in W.

Other classes in W arise similarly. Following Springer [20], we say that an element $w \in W$ is *regular* if w has an eigenvector in $\mathfrak{t}$ whose stabilizer in W is trivial. For example, any power of a Coxeter element is regular. Using Springer's classification of regular elements, along with more recent results of Panyushev [16], one can prove [1])

PROPOSITION 2.2. *Let $w \in W = N/T$ be a regular element of order m. Then*

1. *w has a representative $\sigma \in N$ which has order m and is principal;*
2. *the G-conjugacy class of σ is uniquely determined by the properties in 1.;*
3. *the Kac coordinates of σ (which are well-defined, by 2.) have all $s_i \in \{0,1\}$.*

We call σ the *principal lift* of w. For a given regular element w, there are often several ways to find the Kac coordinates of its principal lift σ. We give just one method, which is not the most efficient, nor can we guarantee that it always works, but it is fun.

As above, let m be the common order of w and σ. A simple argument, using the regularity of w [20, Prop. 4.1], implies that $\langle w\rangle$ permutes the roots in Φ in orbits of size m. Let S be a set of representatives for the $\langle w\rangle$-orbits on Φ. For each $\alpha \in S$, choose a root vector $E_\alpha \in \mathfrak{g}_\alpha$ and let

$$Z_\alpha := E_\alpha + \sigma \cdot E_\alpha + \cdots + \sigma^{m-1} \cdot E_\alpha .$$

The set $\{Z_\alpha : \alpha \in S\}$, along with a basis of $\mathfrak{t}^w$, is a basis of $\mathfrak{g}^\sigma$, so we have the dimension formula

$$\dim \mathfrak{g}^\sigma = \dim \mathfrak{t}^w + \frac{|\Phi|}{m} . \tag{2.14}$$

On the other hand, one can tabulate all possible Kac coordinates for elements of order m, and compute dimensions of centralizers in each case.

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[1]) Some, if not all of this proposition is known to experts, but I could not find complete proofs in the literature. These will appear in forthcoming work of B. Gross, J.-K. Yu and the author.

Let us try this for E_8. Here $|\Phi| = 240$, and the extended Dynkin diagram $\widetilde{\mathcal{D}}(\mathfrak{g})$ has labellings a_i given by

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 4 & 2 \\ & & & & & 3 & & \end{array}$$

There is exactly one regular class in $W(E_8)$ for each order $m \in \{1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30\}$ (see [20, 5.4]). These are precisely the classes in $W(E_8)$ with irreducible minimal polynomials on $\mathfrak{t}$ (cf. [17]) and each nontrivial regular element w has $\mathfrak{t}^w = 0$. For each of the $m > 1$ on this list, we have

$$\dim \mathfrak{g}^\sigma = \frac{240}{m},$$

by equation (2.14). We search through the vectors $(s'_0, \ldots, s'_\ell)$ for which all $s'_i \in \{0, 1\}$ and $\sum_{i=0}^{\ell} a_i s'_i = m$. Each vector corresponds to an automorphism $\sigma' \in G$ of order m and we calculate the dimension of the centralizer $\mathfrak{g}^{\sigma'}$, using (2.7). Remarkably, we find in each case that

$$\dim \mathfrak{g}^{\sigma'} \geq \frac{240}{m},$$

with equality for just one vector $(s_0, \ldots, s_\ell)$, which must then be the Kac coordinates of the principal lift σ of w.

These Kac coordinates have a deeper meaning. If we omit s_0 and double the remaining s_i's, we obtain the weighted Dynkin diagram of another embedding

$$\varphi\colon PGL_2 \hookrightarrow G$$

(see [7] for background). This means that σ lies in this $\varphi(PGL_2)$ as well as the principal PGL_2. The two PGL_2's are conjugate exactly when w is the Coxeter element. The results are tabulated below, using Carter's notations [8] and [7] for Weyl group elements and embeddings $\varphi\colon PGL_2 \hookrightarrow G$, respectively. The first four lines of this table appear in Springer [20, 9.11, 2] (who arrived at them by completely different means). Those entries where $s_0 = 1$ are related to the map between nilpotent elements in $\mathfrak{g}$ and conjugacy classes in W defined by Kazhdan and Lusztig (see [13] and [19]). S. DeBacker informs me that the entries with $s_0 = 0$ are related to a variant of the Kazhdan-Lusztig map. A complete list of Kac coordinates for certain lifts of all Weyl group elements (for E_8 and some smaller groups) can be found in [5].

TABLE 1

Principal lifts of regular elements in $W(E_8)$

Class of $w \in W(E_8)$	$m = \|w\| = \|\sigma\|$	$\dim \mathfrak{g}^\sigma$	Kac coordinates of σ	Class of $PGL_2 \hookrightarrow G$
E_8	30	8	1 1 1 1 1 1 1 1 1	E_8
$E_8(a_1)$	24	10	1 1 1 1 1 0 1 1 1	$E_8(a_1)$
$E_8(a_2)$	20	12	1 1 1 0 1 0 1 1 1	$E_8(a_2)$
$E_8(a_5)$	15	16	1 1 0 1 0 1 0 1 0	$E_8(a_4)$
$E_8(a_3)$	12	20	1 0 1 0 0 1 0 1 0	$E_8(a_5)$
$E_8(a_6)$	10	24	1 0 1 0 0 1 0 0 0	$E_8(a_6)$
$D_8(a_3)$	8	30	0 1 0 0 0 1 0 0 0	$E_8(b_6)$
$E_8(a_8)$	6	40	1 0 0 0 1 0 0 0 0	$E_8(a_7)$
$2A_4$	5	48	0 0 0 0 1 0 0 0 0	$E_8(a_7)$
$2D_4(a_1)$	4	60	0 0 0 1 0 0 0 0 0	$A_4 + A_2$
$4A_2$	3	80	0 0 0 0 0 0 0 0 1	$D_4(a_1) + A_2$
$8A_1$	2	120	0 0 0 0 0 0 0 1 0	$2A_2$

3. SEMISIMPLE AUTOMORPHISMS

We come now to our main purpose, which is to extend Cartan's analysis of inner automorphisms to all torsion automorphisms of $\mathfrak{g}$. Recall that we have fixed a maximal torus and a Borel subgroup $T \subset B$ in $G = \mathrm{Aut}(\mathfrak{g})^\circ$, and $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ is the set of simple roots of T in B.

3.1 Pinned automorphisms

Choose a nonzero vector X_i in the root space $\mathfrak{g}_{\alpha_i}$, for each $1 \leq i \leq \ell$. The triple

$$\mathcal{E} = (T, B, \{X_i\}_{i=1}^{\ell})$$

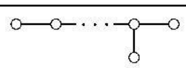
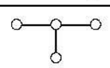
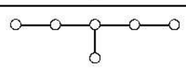
is called a *pinning* (Fr. *épinglage*) and automorphisms of $\mathfrak{g}$ normalizing T, B and permuting $\{X_i\}$ are called *pinned automorphisms*. The group $\mathrm{Aut}(\mathfrak{g}, \mathcal{E})$ of pinned automorphisms is finite and is a complement to G in $\mathrm{Aut}(\mathfrak{g})$:

$$\mathrm{Aut}(\mathfrak{g}) = \mathrm{Aut}(\mathfrak{g}, \mathcal{E}) \ltimes G\,. \tag{3.1}$$

(See [4, VIII. 5.2].) A pinned automorphism can be viewed as a permutation of $\{1, \ldots, \ell\}$ which gives a symmetry of the Dynkin graph $\mathcal{D}(\mathfrak{g})$ of $\mathfrak{g}$. Conversely, for any permutation π of $\{1, \ldots, \ell\}$ giving a symmetry of $\mathcal{D}(\mathfrak{g})$ there is a unique pinned automorphism $\vartheta \in \mathrm{Aut}(\mathfrak{g}, \mathcal{E})$ such that $\vartheta X_i = X_{\pi i}$ for all i. More precisely, for each i there exists $Y_i \in \mathfrak{g}_{-\alpha_i}$, such that the Lie algebra $\mathfrak{g}$ is generated by $\{X_i, Y_i : 1 \leq i \leq \ell\}$ and

$$\vartheta X_i = X_{\pi i}\,, \quad \vartheta Y_i = Y_{\pi i}\,.$$

Thus, $\mathrm{Aut}(\mathfrak{g}, \mathcal{E})$ is isomorphic to the symmetry group of $\mathcal{D}(\mathfrak{g})$, hence has order six when $\mathfrak{g}$ has type D_4, order two in types $A_n, D_n\,(n \geq 5)$ and E_6, and is trivial otherwise. The nontrivial pinned automorphisms and their fixed-point subalgebras are tabulated as follows.

Type	$\mathfrak{g}$	$\mathcal{D}(\mathfrak{g})$	$f = \lvert\vartheta\rvert$	$\mathcal{D}(\mathfrak{g}^{\vartheta})$	$\mathfrak{g}^{\vartheta}$
${}^2A_{2n}$ $(n \geq 1)$	$\mathfrak{sl}_{2n+1}$	o—o—···—o—o···—o—o	2	o—o—···—o$\Rightarrow$o	$\mathfrak{so}_{2n+1}$
${}^2A_{2n-1}$ $(n \geq 2)$	$\mathfrak{sl}_{2n}$	o—o—···o···—o—o	2	o—o—···—o$\Leftarrow$o	$\mathfrak{sp}_{2n}$
${}^2D_{n+1}$	$\mathfrak{so}_{2n+2}$		2	o—o—···—o$\Rightarrow$o	$\mathfrak{so}_{2n+1}$
${}^3D_4$	$\mathfrak{so}_8$		3	o$\Lleftarrow$o	$\mathfrak{g}_2$
${}^2E_6$	$\mathfrak{e}_6$		2	o—o$\Leftarrow$o—o	$\mathfrak{f}_4$

Fix now a pinned automorphism $\vartheta \in \mathrm{Aut}(\mathfrak{g}, \mathcal{E})$ of order f and denote also by ϑ the permutation of $\{1, \ldots, \ell\}$ which it induces. We have $f \in \{1, 2, 3\}$, and $f = 1$ reduces to the inner case treated above. Let I be the set of orbits in $\{1,\dots,\ell\}$ under ϑ. The fixed-point algebra $\mathfrak{g}^\vartheta$ is simple and is generated by the elements

$$X_\iota = \sum_{i\in\iota} X_i\,,\quad Y_\iota = \sum_{i\in\iota} Y_i\,,\quad \text{for } \iota \in I$$

(see [10, X.5]).

LEMMA 3.1. *For any $\vartheta \in \mathrm{Aut}(\mathfrak{g},\mathcal{E})$, the fixed-point groups T^ϑ and G^ϑ are connected and T^ϑ is a maximal torus in G^ϑ. If ϑ is nontrivial then G^ϑ is equal to the full automorphism group* $\mathrm{Aut}(\mathfrak{g}^\vartheta)$.

Proof. Since ϑ permutes the basis $\{\check\omega_1,\dots,\check\omega_\ell\}$ of $X_*(T)$, it follows that T^ϑ is connected, of dimension equal to the number of ϑ-orbits on this basis. Let $G' \to G$ be the simply-connected covering of G with kernel $Z' = \pi_1(G)$ and let T' be the pre-image of T in G'. The set $M = \{\omega_i \,:\, a_i = 1\}$ of minuscule weights of T' restricts bijectively to the character group of Z', so that the ϑ-invariant elements of M are the characters of $Z'/(1-\vartheta)Z'$. It follows that the map

$$Z'/(1-\vartheta)Z' \longrightarrow T'/(1-\vartheta)T'$$

induced by the inclusion $Z' \hookrightarrow T'$ is injective. The connectedness of G^ϑ now follows from [21, 9.3, 9.5].

Since $X_1 + \cdots + X_\ell$ is a regular nilpotent element in $\mathfrak{g}$ contained in $\mathfrak{g}^\vartheta$, there is a principal PGL_2 in G contained in G^ϑ. It follows that T^ϑ contains regular elements in G. Since the centralizer of a principal PGL_2 in G is trivial, it follows that G^ϑ has trivial center.

The nodes of the Dynkin graph $\mathcal{D}(\mathfrak{g}^\vartheta)$ correspond to the ϑ-orbits on $\{1,\dots,\ell\}$ and from the table above, we see that $\mathcal{D}(\mathfrak{g}^\vartheta)$ has trivial symmetry group. Hence $\mathrm{Aut}(\mathfrak{g}^\vartheta)$ is connected and $G^\vartheta = \mathrm{Aut}(\mathfrak{g}^\vartheta)$.

3.2 CONJUGACY RESULTS

The first step in the classification of semisimple inner automorphisms was the fact that T meets every semisimple conjugacy class in G. In the outer case, we begin with an analogous result.

LEMMA 3.2. *Every semisimple automorphism σ of $\mathfrak{g}$ is G-congugate to one of the form ϑs, where $\vartheta \in \mathrm{Aut}(\mathfrak{g},\mathcal{E})$ and $s \in T^\vartheta$.*

Proof. From [21, Thm. 7.5], σ preserves a Borel subgroup of G and a maximal torus therein. Replacing σ by a G-conjugate, we may assume that these are B and T, respectively. Let ϑ be the projection of σ in $\mathrm{Aut}(\mathfrak{g},\mathcal{E})$ according to (3.1). So $\sigma = \vartheta s'$, for some $s' \in G$. Since ϑ preserves T, B, the element s' normalizes T, B. Hence s' projects to an element w of the Weyl group of T which preserves the set of simple roots Δ determined by B. This means that $w = 1$, so $s' \in T$.

Let $p\colon T^{\vartheta} \to T/(1-\vartheta)T$ be the restriction to T^{ϑ} of the natural projection $T \to T/(1-\vartheta)T$. The kernel

$$\ker p = T^{\vartheta} \cap (1-\vartheta)T$$

is finite. Indeed, if f is the order of ϑ then the mapping $t \mapsto t\cdot\vartheta(t)\cdots\vartheta^{f-1}(t)$ sends $(1-\vartheta)T$ to 1 and sends every element of T^{ϑ} to its f^{th} power. It follows that $\ker p$ is contained in the f-torsion subgroup of T, hence $\ker p$ is finite of order dividing f^{ℓ}. Since T^{ϑ} and $T/(1-\vartheta)T$ have the same dimension, it follows that p is surjective. Hence there is $t \in T$ such that

$$\vartheta^{-1}(t)s't^{-1} \in T^{\vartheta}.$$

Conjugating in $\mathrm{Aut}(\mathfrak{g})$, we have

$$t\sigma t^{-1} = t\vartheta s't^{-1} = \vartheta\cdot\vartheta^{-1}(t)s't^{-1} \in \vartheta T^{\vartheta},$$

as claimed.

Thus, any element of ϑG is G-conjugate to one of the form $\sigma = \vartheta s$, with $s \in T^{\vartheta}$. As a partial step towards torsion automorphisms, we will first restrict s to lie in S^{ϑ}, where $S := \exp(V)$ is the maximal compact subgroup of T. The conjugation action of G on ϑG induces actions of W^{ϑ} and S on ϑS, hence an action of $W^{\vartheta} \ltimes S$ on ϑS.

LEMMA 3.3. *If two elements of ϑS are G-conjugate, then they are conjugate under $W^{\vartheta} \ltimes S$.*

Proof. Suppose $s, s' \in S$ and $g \in G$ are such that $g\vartheta s g^{-1} = \vartheta s'$. Writing $g^{\vartheta} := \vartheta^{-1}(g)$, this means that

$$g^{\vartheta}\cdot s = s'\cdot g\,.$$

For the moment we care only that $s, s' \in T$. Following the argument in [1, Lemma 6.5], we will show that ϑs and $\vartheta s'$ are conjugate under $N^{\vartheta}\cdot T$. Using the Bruhat decomposition for G, there is a unique $n \in N$ such that $g = unv$, with u, v in the unipotent radical U of our ϑ-stable Borel subgroup B.

We can replace g by n. Indeed, we have

$$u^\vartheta n^\vartheta v^\vartheta \cdot s = s' \cdot unv\,.$$

Writing both sides in the form UNU and comparing the parts in N on both sides, we find that

$$n^\vartheta \cdot s = s' \cdot n\,. \tag{3.2}$$

This shows that $n(\vartheta s)n^{-1} = \vartheta s'$, as claimed, and also that the image of n in W belongs to W^ϑ. We have already remarked that every element $w \in W^\vartheta$ has a representative $\dot{w} \in N^\vartheta$. Hence $n = \dot{w}t$, for some $t \in T$, so ϑs and $\vartheta s'$ are $N^\vartheta \cdot T$-conjugate, as claimed.

Now suppose $s, s' \in S$. Using the polar decomposition $T = S \times H$, where $H \simeq (\mathbf{R}_{>0})^\ell$, we write $t = t_c t_h$, with $t_c \in S$ and $t_h \in H$. From equation (3.2) we have

$$(s')^w = (t_c t_h)^\vartheta \cdot s \cdot (t_c t_h)^{-1} = (t_c^\vartheta t_c^{-1} s) \cdot t_h^\vartheta t_h^{-1}.$$

Since both $(s')^w$ and $t_c^\vartheta t_c^{-1} s$ belong to S, it follows that $t_h^\vartheta = t_h$, and that

$$n_c^\vartheta \cdot s = s' \cdot n_c\,,$$

where $n_c = \dot{w}t_c \in N^\vartheta \cdot S$. Since the action of N^ϑ on ϑS factors through $N^\vartheta / T^\vartheta = W^\vartheta$, the lemma is proved.

To study S-conjugacy on ϑS^ϑ, we linearize as follows. Our pinned automorphism ϑ permutes the basis $\{\check{\omega}_i\}$ of Y. Let

$$P_\vartheta = f^{-1}(1 + \vartheta + \cdots + \vartheta^{f-1}) \in \operatorname{End}(V)$$

be the projection onto V^ϑ and set

$$Y_\vartheta = P_\vartheta Y\,.$$

Then Y_ϑ is a lattice in V^ϑ, and contains the group Y^ϑ of ϑ-invariants in Y as a (generally proper) sublattice.

LEMMA 3.4. *Let $x, x' \in V^\vartheta$. Then $\vartheta \exp(x)$ and $\vartheta \exp(x')$ are S-conjugate if and only if $x - x' \in Y_\vartheta$.*

Proof. A straightforward calculation shows that

$$\exp(-v) \cdot \vartheta \exp(x) \cdot \exp(v) = \vartheta \exp(x')$$

for some $v \in V$ if and only if

$$x - x' \in [(1 - \vartheta)V + Y] \cap V^\vartheta\,.$$

We show that

$$[(1-\vartheta)V + Y] \cap V^\vartheta = Y_\vartheta \,. \tag{3.3}$$

Since P_ϑ kills $(1-\vartheta)V$ and is the identity map on V^ϑ, the left side of (3.3) is contained in the right side. The reverse containment follows from the fact that the polynomial

$$p(x) = f^{-1}(1 + x + x^2 + \cdots + x^{f-1})$$

satisfies the differential equation $p(x) = (1-x)p'(x) + x^{f-1}$.

The W^ϑ-action on V^ϑ extends to an affine action of the group

$$\widetilde{W}_\vartheta := W^\vartheta \ltimes Y_\vartheta \,,$$

where $y \in Y_\vartheta$ acts by the translation $t_y \colon x \mapsto x + y$. Lemmas 3.3 and 3.4 combine to yield

LEMMA 3.5. *Let $x, x' \in V^\vartheta$. Then $\vartheta\exp(x)$ and $\vartheta\exp(x')$ are G-conjugate if and only if x and x' belong to the same $\widetilde{W}_\vartheta$-orbit on V^ϑ.*

Since $\exp(x)$ is torsion if and only if $x \in V_{\mathbf{Q}} := \mathbf{Q} \otimes Y$, Lemma 3.5 implies

COROLLARY 3.6. *The map $x \mapsto \vartheta\exp(x)$ induces a bijection between the set of $\widetilde{W}_\vartheta$-orbits on $V^\vartheta_{\mathbf{Q}}$ and the set of G-conjugacy classes of torsion elements in ϑG.*

3.3 A FUNDAMENTAL DOMAIN FOR $\widetilde{W}_\vartheta$ IN V^ϑ AND KAC COORDINATES

We shall use the geometry of the $\widetilde{W}_\vartheta$-action on V^ϑ to recover Kac's parametrization of the G-conjugacy classes of torsion elements in ϑG. Throughout this section it may help the reader to look ahead at Table 2 and Section 4, where the individual cases are treated in detail.

Recall that I denotes the set of orbits in $\{1, \ldots, \ell\}$ under the permutation induced by the action of ϑ on the set $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ of simple roots. For $\iota \in I$, let w_ι be the unique element in the subgroup of W generated by the reflections $\{r_i : i \in \iota\}$ such that $\{w_\iota \alpha_i : i \in \iota\} = \{-\alpha_i : i \in \iota\}$. Then W^ϑ is a Coxeter group with generators $\{w_\iota : \iota \in I\}$ and V^ϑ is the reflection representation of W^ϑ (see [21]). The lattice Y_ϑ has the $\mathbf{Z}$-basis $\{\check{\mu}_\iota : \iota \in I\}$, where $\check{\mu}_\iota = P_\vartheta(\check{\omega}_i)$ for any $i \in \iota$. That is,

$$\check{\mu}_\iota = \frac{1}{|\iota|}\sum_{i\in\iota}\check{\omega}_i ,$$

where $|\iota|$ denotes the cardinality of the ϑ-orbit ι. The action of $\widetilde{W}_\vartheta$ on V^ϑ is generated by the reflections w_ι and the translations by $\check{\mu}_\iota$. As the notation indicates, $\widetilde{W}_\vartheta$ is an extended affine Weyl group, of a root system Φ_ϑ defined as follows. Say that two roots α, β in Φ are *ϑ-equivalent* if their restrictions $\bar{\alpha}$ and $\bar{\beta}$ to V^ϑ are positively proportional: $\bar{\alpha} = r\bar{\beta}$ for some $r > 0$. A ϑ-equivalence class $a \subset \Phi$ can have one of two types:

I. $a = \{\alpha, \vartheta\alpha, \dots\}$ is a ϑ-orbit consisting of mutually orthogonal roots;

II. $a = \{\alpha, \vartheta\alpha, \alpha + \vartheta\alpha\}$, occurring only in type ${}^2A_{2n}$.

Let Φ/ϑ denote the set of ϑ-equivalence classes of roots in Φ. For each $a \in \Phi/\vartheta$, set

$$\gamma_a := \sum_{\alpha\in a}\bar{\alpha}, \quad \text{and} \quad \Phi_\vartheta := \{\gamma_a : a \in \Phi/\vartheta\} .$$

Then Φ_ϑ is a reduced root system and $\widetilde{W}_\vartheta$ is its extended affine Weyl group. Note that γ_a is generally not the restriction to V^ϑ of a root in a. If we choose $\alpha \in a$ as in the definitions of types I and II, i.e., so that $\bar{\alpha}$ is not twice the restriction of another root in a, and set $\beta_a = \bar{\alpha}$, then $\gamma_a = f_a\beta_a$, where

$$f_a = \begin{cases} |a| & \text{in type I}, \\ 4 & \text{in type II}. \end{cases}$$

A base Δ_ϑ of Φ_ϑ is obtained from the base Δ of Φ as follows. Given a ϑ-orbit $\iota \in I$, let $a_\iota \in \Phi/\vartheta$ denote the unique ϑ-equivalence class containing $\{\alpha_i : i \in \iota\}$, and set

$$\gamma_\iota := \gamma_{a_\iota}, \quad f_\iota := f_{a_\iota} .$$

Then $\Delta_\vartheta := \{\gamma_\iota : \iota \in I\}$ is a base of Φ_ϑ; we have $\langle \gamma_\iota, \check{\mu}_{\iota'}\rangle = 0$ if $\iota \neq \iota'$ and

$$\langle \gamma_\iota, \check{\mu}_\iota\rangle = \frac{f_\iota}{|\iota|} = \begin{cases} 1 & \text{if } a_\iota \text{ has type I}, \\ 2 & \text{if } a_\iota \text{ has type II}. \end{cases} \tag{3.4}$$

The equations $\gamma_a = n$, for $\gamma_a \in \Phi_\vartheta$ and $n \in \mathbf{Z}$, give hyperplanes in V^ϑ and the complement of all these hyperplanes is a union of alcoves which are permuted transitively by the group $\widetilde{W}_\vartheta = W^\vartheta \ltimes Y_\vartheta$. Outside of ${}^2A_{2n}$ this follows from (3.4), which shows that the $\check{\mu}_\iota$ are the fundamental weights for Δ_ϑ. For ${}^2A_{2n}$ see Section 4.1.

Just as in the inner case, the base Δ_ϑ determines an alcove C_ϑ in V^ϑ, as follows. Let $\widetilde{\gamma}_0$ be the highest root of Φ_ϑ with respect to the base Δ_ϑ. We obtain positive integers c_ι, for $\iota \in I$, defined as

$$\widetilde{\gamma}_0 = \sum_{\iota \in I} c_\iota \gamma_\iota .$$

The integers c_ι are found in Table 2 below. As in the inner case, we set

$$\widetilde{I} := \{0\} \cup I, \quad \gamma_0 := 1 - \widetilde{\gamma}_0, \quad c_0 = 1,$$

so that

$$\sum_{\iota \in \widetilde{I}} c_\iota \gamma_\iota \equiv 1$$

on V^ϑ, and our alcove is defined by

$$C_\vartheta := \{x \in V^\vartheta \,:\, \langle \gamma_\iota, x \rangle > 0 \quad \forall \iota \in \widetilde{I}\}.$$

Note that C_ϑ is not equal to $C \cap V^\vartheta$, in general. The set of vertices of C_ϑ is $\{v_\iota \,:\, \iota \in \widetilde{I}\}$, where

$$v_0 = \check{\mu}_0 = 0 \quad \text{and} \quad v_\iota = \langle \widetilde{\gamma}_0, \check{\mu}_\iota \rangle^{-1} \check{\mu}_\iota , \quad \text{for} \quad \iota \in I.$$

A point $x \in \overline{C}_\vartheta$ may be uniquely expressed in barycentric coordinates as

$$x = \sum_{\iota \in \widetilde{I}} x_\iota v_\iota , \quad \text{with} \quad \sum_{\iota \in \widetilde{I}} x_\iota = 1 \quad \text{and} \quad x_\iota \geq 0 \quad \forall \iota \in \widetilde{I}.$$

As in the inner case, any point in V^ϑ is $\widetilde{W}_\vartheta$-conjugate to a point in $\overline{C}_\vartheta$ and two points in $\overline{C}_\vartheta$ are conjugate under $\widetilde{W}_\vartheta$ if and only if they are conjugate under the alcove stabilizer

$$\Omega_\vartheta := \{\rho \in \widetilde{W}_\vartheta \,:\, \rho \cdot C_\vartheta = C_\vartheta\}.$$

The action of each element $\rho \in \Omega_\vartheta$ on C_ϑ is given in barycentric coordinates as a permutation of $\widetilde{I}$, via the action of ρ on the vertices of C_ϑ.

We recover the Kac classification by taking a closer look at the vertices $v_\iota = \langle \widetilde{\gamma}_0, \check{\mu}_\iota \rangle^{-1} \check{\mu}_\iota$. From (3.4), we have

$$\langle \widetilde{\gamma}_0, \check{\mu}_\iota \rangle = \frac{f_\iota c_\iota}{|\iota|} . \tag{3.5}$$

I claim that

$$f \text{ divides } f_\iota c_\iota \quad \text{for all} \quad \iota \in I. \tag{3.6}$$

This is clear if $f = |\iota|$. Otherwise, we are not in type ${}^2A_{2n}$ and since f is a prime, the orbit $\iota = \{i\}$ is a singleton. Being the highest root of Φ_ϑ, $\widetilde{\gamma}_0$ is a long root, hence it is the sum of a ϑ-equivalence class (in fact a ϑ-orbit) $a_0 = \{\alpha_1, \dots, \alpha_f\}$ of cardinality f. From (3.5) we have

$$f_\iota c_\iota = \langle \widetilde{\gamma}_0, \check{\mu}_\iota \rangle = \langle \alpha_1 + \cdots + \alpha_f, \check{\omega}_i \rangle = f \langle \alpha_1, \check{\omega}_i \rangle ,$$

which is divisible by f, as claimed. If we set

$$f_0 = f ,$$

then (3.6) also holds for $\iota = 0$. Thus, we have integers

$$b_\iota := \frac{f_\iota c_\iota}{f} \quad \text{for } \iota \in \widetilde{I}, \quad \text{with } b_0 = 1 . \tag{3.7}$$

We can now state the Kac classification of torsion elements in ϑG.

THEOREM 3.7. *The G-conjugacy classes of torsion elements in ϑG are classified as follows.*

1. *Every torsion element in ϑG is G-conjugate to one of the form $\sigma = \vartheta \exp(x)$, where $x \in \overline{C}_\vartheta \cap V_{\mathbf{Q}}$.*

2. *There are nonnegative integers s_ι, indexed by $\iota \in \widetilde{I}$, such that $\gcd\{s_\iota : \iota \in \widetilde{I}\} = 1$, the order m of σ is given by*

$$m = f \cdot \sum_{\iota \in \widetilde{I}} b_\iota s_\iota ,$$

and x is given in barycentric coordinates as

$$x = \frac{f}{m} \cdot \sum_{\iota \in \widetilde{I}} b_\iota s_\iota v_\iota .$$

3. *Two torsion automorphisms σ, σ', with coordinates (s_ι) and (s'_ι), are G-conjugate if and only if there is a permutation ρ of $\widetilde{I}$ arising from Ω_ϑ such that $s'_\iota = s_{\rho\iota}$ for all $\iota \in \widetilde{I}$.*

Proof. The assertions in parts 1. and 3. are immediate from the above discussion and Corollary 3.6. Since $\exp(x)$ and ϑ commute, the order m of $\sigma = \vartheta \exp(x)$ is divisible by f and we have $\exp(mx) = 1$. Hence there are integers $s_1, s_2, \dots, s_\ell$ such that

$$x = \frac{1}{m} \sum_{i=1}^{\ell} s_i \check{\omega}_i .$$

Since x is ϑ-fixed, each s_i depends only on the ϑ-orbit ι containing i; we write $s_\iota := s_i$ for $i \in \iota$, and we have

$$x = \frac{1}{m}\sum_{\iota \in I} |\iota|\, s_\iota \check{\mu}_\iota . \tag{3.8}$$

Since $x \in \overline{C}_\vartheta$, we have

$$1 \geq \langle \widetilde{\gamma}_0, x\rangle = \frac{1}{m}\sum_{\iota\in I} s_\iota\, |\iota|\, \langle \widetilde{\gamma}_0, \check{\mu}_\iota\rangle = \frac{1}{m}\sum_{\iota\in I} s_\iota c_\iota f_\iota = \frac{f}{m}\sum_{\iota\in I} b_\iota s_\iota .$$

We define a nonnegative integer s_0 by

$$s_0 = \frac{m}{f} - \sum_{\iota\in I} b_\iota s_\iota ,$$

so that

$$f \cdot \sum_{\iota\in\widetilde{I}} b_\iota s_\iota = m .$$

If d divides s_ι for all $\iota \in \widetilde{I}$, then d divides m/f, so f divides m/d and we have

$$\vartheta^{m/d} = 1 = \exp\left(mx/d\right) ,$$

implying that $\sigma^{m/d} = 1$. Therefore $d = 1$ and the integers s_ι are relatively prime.

From (3.8) and (3.5), the integers (s_ι) are related to the barycentric coordinates (x_ι) of x by

$$\frac{|\iota|\, s_\iota}{m} = \frac{x_\iota}{\langle\widetilde{\gamma}_0, \mu_\iota\rangle} = \frac{|\iota|\, x_\iota}{c_\iota f_\iota} = \frac{|\iota|\, x_\iota}{f\, b_\iota} ,$$

or

$$x_\iota = \frac{f}{m}\cdot b_\iota s_\iota .$$

This shows that all s_ι are nonnegative and completes the proof of part 2.

REMARK. The integers $(s_\iota)_{\iota\in\widetilde{I}}$ are the *Kac coordinates* of σ (cf. [12, Thm. 8.5]). The integers b_ι are the labels of Kac's twisted affine diagrams, as we will see in the next section.

3.4 FIXED-POINT SUBALGEBRAS

In this section we determine the subalgebra $\mathfrak{g}^\sigma$ fixed by a torsion automorphism $\sigma \in \vartheta G$, in terms of the geometry of the alcove C_ϑ. The first step is to compute the matrix of ϑ acting on $\mathfrak{g}$. For each ϑ-equivalence class $a \in \Phi/\vartheta$, the direct sum

$$\mathfrak{g}_a = \sum_{\alpha \in a} \mathfrak{g}_\alpha$$

is preserved by ϑ, the root spaces being permuted. I claim that if $a = \{\alpha, \vartheta\alpha, \dots\}$ has type I, then ϑ acts on $\mathfrak{g}_a$ via the permutation matrix of an f_a-cycle. If $f_a > 1$, and we choose any nonzero $X_\alpha \in \mathfrak{g}_\alpha$, then $\{X_\alpha, \vartheta X_\alpha, \dots\}$ is a basis of $\mathfrak{g}_a$ permuted by ϑ. If $a = \{\alpha\}$ has type I with $f_a = 1$ then we can find $w \in W^\vartheta$ and $\iota \in I$ such that $\alpha = w\beta$, where $\beta \in \Delta$. By [21] we can choose a lift $n \in N^\vartheta$ of w so that $\mathrm{Ad}(n)\colon \mathfrak{g}_\beta \to \mathfrak{g}_\alpha$ is ϑ-equivariant. By definition, the pinned automorphism ϑ fixes $\mathfrak{g}_\beta$ pointwise, so ϑ also fixes $\mathfrak{g}_\alpha$ pointwise, as claimed.

If $a = \{\alpha, \vartheta\alpha, \alpha + \vartheta\alpha\}$ has type II, and we again choose any nonzero $X_\alpha \in \mathfrak{g}_\alpha$, then $(X_\alpha, \vartheta X_\alpha, [X_\alpha, \vartheta X_\alpha])$ is an ordered basis of $\mathfrak{g}_a$ on which ϑ has matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Now let $\sigma = \vartheta s$, where $s \in T^\vartheta$. The characteristic polynomial of σ on $\mathfrak{g}_a$ is given as follows. Recall from the previous section that we defined β_a to be the shortest restriction to V^ϑ of a root in a, and that we have

$$\gamma_a = f_a \beta_a .$$

Our matrix calculations show that

$$\det(t - \sigma|\mathfrak{g}_a) = \begin{cases} t^{f_a} - \beta_a(s)^{f_a} & \text{if } a \text{ has type I}, \\ (t^2 - \beta_a(s)^2)(t + \beta_a(s)^2) & \text{if } a \text{ has type II}. \end{cases} \tag{3.9}$$

In all cases, the roots of $\det(t - \sigma|\mathfrak{g}_a)$ are distinct and we have

$$\det(1 - \sigma|\mathfrak{g}_a) = 1 - \beta_a(s)^{f_a} = 1 - \gamma_a(s).$$

If $s = \exp(x)$ with $x \in V^\vartheta$ this means that

$$\dim \mathfrak{g}_a^\sigma = \begin{cases} 1 & \text{if } \langle \gamma_a, x\rangle \in \mathbf{Z}, \\ 0 & \text{if } \langle \gamma_a, x\rangle \notin \mathbf{Z}. \end{cases} \tag{3.10}$$

Thus, the integrality of $\langle \gamma_a, x\rangle$ determines when $\mathfrak{g}_a^\sigma$ is nonzero. However, the root γ_a is not the character of T^ϑ on $\mathfrak{g}_a^\sigma$. Indeed, if $\langle \gamma_a, x\rangle \in \mathbf{Z}$, the matrix calculations above show that the line $\mathfrak{g}_a^\sigma$ affords the character β_a or $2\beta_a$, the latter occurring if and only if a has type II and $\langle \gamma_a, x\rangle$ is odd.

As we saw for $\vartheta = 1$, the closure

$$\overline{C}_\vartheta = \{x \in V^\vartheta \ : \ \gamma_\iota \geq 0 \quad \forall \iota \in \widetilde{I}\}$$

is partitioned into a disjoint union of $2^{|\widetilde{I}|} - 1$ facets:

$$\overline{C}_\vartheta = \bigcup_{J \subsetneq \widetilde{I}} C_\vartheta^J ,$$

indexed by the subsets $J \subset \widetilde{I}$ with $J \neq \widetilde{I}$. The facet C_ϑ^J consists of the points $x \in \overline{C}_\vartheta$ such that $\langle \gamma_\iota, x\rangle = 0$ for $\iota \in J$ and $\langle \gamma_\iota, x\rangle > 0$ for $\iota \in \widetilde{I} - J$. For a general root $\gamma_a \in \Phi_\vartheta$, we have $\langle \gamma_a, x\rangle \in \mathbf{Z}$ if and only if $\langle \gamma_a, x\rangle \in \{-1, 0, 1\}$. Thus, we have proved:

PROPOSITION 3.8. *If $x \in C_\vartheta^J$ and $\sigma = \vartheta \exp(x)$, then we have the root-space decomposition*

$$\mathfrak{g}^\sigma = \mathfrak{t}^\vartheta \ominus \sum_a \mathfrak{g}_a^\sigma ,$$

where the sum is over those ϑ-equivalence classes $a \in \Phi/\vartheta$ for which $\langle \gamma_a, x\rangle \in \{-1, 0, 1\}$. Each such $\mathfrak{g}_a^\sigma$ is a one-dimensional eigenspace for T^ϑ, affording either the root β_a or $2\beta_a$, the latter occurring if and only if a has type II *and $\langle \gamma_a, x\rangle = \pm 1$.*

The root $2\beta_a$ appears only in the case ${}^2A_{2n}$; for more details in this case see Section 4.1.

Taking $x = 0$, Proposition 3.8 says that $\mathfrak{g}^\vartheta$ has root system

$$\widehat{\Phi}_\vartheta := \{\beta_a : a \in \Phi/\vartheta\}, \quad \text{with base} \quad \widehat{\Delta}_\vartheta := \{\beta_\iota : \iota \in I\},$$

where $\beta_\iota = f_\iota^{-1}\gamma_\iota$. If we set $\widetilde{\beta}_0 = f^{-1}\widetilde{\gamma}_0$, then

$$\widetilde{\beta}_0 = \sum_{\iota \in I} b_\iota \beta_\iota , \tag{3.11}$$

where the integers $b_\iota = c_\iota f_\iota / f$ are the ones previously arrived at in (3.7). For (G, ϑ) not of type ${}^2A_{2n}$, the root $\widetilde{\beta}_0$ is the highest short root of $\widehat{\Phi}_\vartheta$. For ${}^2A_{2n}$, $\widetilde{\beta}_0$ is twice the highest short root of $\widehat{\Phi}_\vartheta$ (see 4.1). In all cases, we set $\beta_0 = f^{-1}\gamma_0$, and recall that $b_0 = 1$, so that

$$\sum_{\iota \in \widetilde{I}} b_\iota \beta_\iota \equiv \frac{1}{f} .$$

To complete the picture, we must also give the co-roots of $\widehat{\Phi}_\vartheta$. For every ϑ-equivalence class $a \in \Phi/\vartheta$, the co-root $\check{\beta}_a$ is defined by

$$\check{\beta}_a = \sum_{\alpha \in a} \check{\alpha}\,, \tag{3.12}$$

which makes $\langle \beta_a, \check{\beta}_a \rangle = 2$, and we set $\check{\beta}_\iota = \check{\beta}_{a_\iota}$, for $\iota \in I$. Then the reflection $w_\iota \in W_\vartheta$ about the hyperplane $\beta_\iota = 0$ is given by

$$w_\iota \cdot x = x - \langle \beta_\iota, x \rangle \check{\beta}_\iota\,, \quad \text{for all } \iota \in I\,. \tag{3.13}$$

We also define $\check{\beta}_0$ so that (3.13) also holds for $\iota = 0$. Recall that a_0 is the ϑ-equivalence class such that $\widetilde{\beta}_0 = \beta_{a_0}$. Our $\check{\beta}_0$ will be a multiple of $\check{\beta}_{a_0}$, where the latter has been defined in (3.12). To make (3.13) hold, we must take

$$\check{\beta}_0 = \begin{cases} -\frac{1}{2}\check{\beta}_{a_0} & \text{in type } {}^2A_{2n}\,, \\ -\check{\beta}_{a_0} & \text{in all other types}\,. \end{cases}$$

All of this data is displayed in the *Kac diagram* $\mathcal{D}(\mathfrak{g}, \vartheta)$, which has nodes indexed by $\iota \in \widetilde{I}$ and labelled by the integers b_ι, for $\iota \in \widetilde{I}$; the number of bonds between nodes ι, κ is the integer

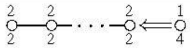
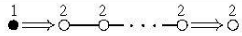
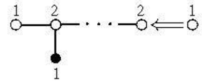
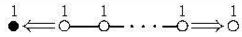
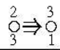
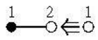
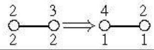
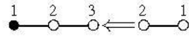
$$n_{\iota,\kappa} := \langle \beta_\iota, \check{\beta}_\kappa \rangle \cdot \langle \beta_\kappa, \check{\beta}_\iota \rangle \in \{0, 1, 2, 3, 4\}\,.$$

We get $n_{\iota,\kappa} = 4$ only in type ${}^2A_2$ (see Section 4.1), and we get $n_{\iota,\kappa} = 3$ only in type ${}^3D_4$ (see 4.4). If $n_{\iota,\kappa} \geq 2$, then we may order ι, κ so that $\langle \beta_\iota, \check{\beta}_\kappa \rangle = -1$ and $\langle \beta_\kappa, \check{\beta}_\iota \rangle = -n_{\iota,\kappa}$. Then on the bonds we put an arrow pointing towards ι, as in the following example:

$$\underset{\iota}{\circ} \Lleftarrow \underset{\kappa}{\circ} \quad \text{means} \quad \langle \beta_\iota, \check{\beta}_\kappa \rangle = -1\,, \quad \langle \beta_\kappa, \check{\beta}_\iota \rangle = -3\,.$$

The Kac diagrams appear in the fourth column of Table 2. For any $x \in \overline{C}_\vartheta$, deleting from $\mathcal{D}(\mathfrak{g}, \vartheta)$ the nodes ι for which $\langle \gamma_\iota, x \rangle \in \mathbf{Z}$ gives the Dynkin diagram $\mathcal{D}(\mathfrak{g}^\sigma)$, by Proposition 3.8. We denote the node corresponding to β_0 by $\bullet$; deleting just this node gives the Dynkin diagram $\mathcal{D}(\mathfrak{g}^\vartheta)$. Above each node of $\mathcal{D}(\mathfrak{g}, \vartheta)$, we give the integers b_ι. These integers are denoted by Kac as a_i in [12, Chap. 8]; he arrived at them, along with his diagrams $\mathcal{D}(\mathfrak{g}, \vartheta)$, in a completely different way.

TABLE 2
Root systems Φ_ϑ, $\widehat{\Phi}_\vartheta$ and Kac diagrams $\mathcal{D}(\mathfrak{g},\vartheta)$

Type	Φ_ϑ	c_ι (Φ_ϑ-diagram) f_ι	b_ι $\mathcal{D}(\mathfrak{g},\vartheta)$	$\widehat{\Phi}_\vartheta$	$\lvert\Omega_\vartheta\rvert$
${}^2A_2$	C_1			B_1	1
${}^2A_{2n}$ ($n\geq 2$)	C_n			B_n	1
${}^2A_{2n-1}$ ($n\geq 3$)	B_n			C_n	2
${}^2D_{n+1}$ ($n\geq 2$)	C_n			B_n	2
${}^3D_4$	G_2			$\widehat{G}_2$	1
${}^2E_6$	F_4			$\widehat{F}_4$	1

On the left side of Table 2, we also give the unextended diagrams of the root systems Φ_ϑ, along with the integers c_ι and f_ι above and below each node, respectively. Recall that these numbers were used to compute the b_ι's, via the relation $c_\iota f_\iota = b_\iota f$. The rightmost column of Table 2 gives the alcove stabilizer Ω_ϑ, discussed in Section 3.6 below.

3.5 COMPUTING KAC COORDINATES IN THE OUTER CASE

As we saw for $\vartheta = 1$, the subgroup $\widetilde{W}^\circ_\vartheta$ of $\widetilde{W}_\vartheta$ generated by the reflections w_ι, for $\iota \in \widetilde{I}$, is the affine Weyl group of the root system Φ_ϑ, and the alcoves in V^ϑ are permuted simply transitively by $\widetilde{W}^\circ_\vartheta$. From the formula

$$w_\iota \cdot x = x - \langle\beta_\iota, x\rangle\check{\beta}_\iota, \quad \text{for } x \in V^\vartheta \text{ and } \iota \in \widetilde{I},$$

we can express the action of $\widetilde{W}^\circ_\vartheta$ on V^ϑ in terms of Kac coordinates, just as we did in Section 2.3: if $x \in V^\vartheta$ has barycentric coordinates $(s_\iota)_{\iota\in\widetilde{I}}$, where some of the s_ι's may be negative, then $w_\kappa \cdot x$ has barycentric coordinates $(s'_\iota)_{\iota\in\widetilde{I}}$, where

$$s'_\iota = s_\iota - \langle\beta_\iota, \check{\beta}_\kappa\rangle s_\kappa .$$

The algorithm for conjugating x into $\overline{C}_\vartheta$ runs just as in Section 2.3. Thus the diagram $\mathcal{D}(\mathfrak{g},\vartheta)$ contains instructions for finding the Kac coordinates of the automorphism $\vartheta\exp(x)$, where x is any rational point in V^ϑ.

3.6 THE COMPONENT GROUP OF G^σ

Let $x \in \overline{C}_\vartheta$, with $\sigma = \vartheta \cdot \exp(x)$ as before. Lemma 3.8 determines the connected centralizer $(G^\sigma)^\circ$ up to isogeny, in terms of the facet C_ϑ^J containing x. As in case $\vartheta = 1$, the component group A_σ of G^σ depends on the location of x in C_ϑ^J, and is governed by the alcove stabilizer

$$\Omega_\vartheta = \{\rho \in \widetilde{W}_\vartheta \ : \ \rho \cdot C_\vartheta = C_\vartheta\}.$$

More precisely, we have:

LEMMA 3.9. *If $\sigma = \vartheta \exp(x)$, with $x \in \overline{C}_\vartheta$, then $A_\sigma \simeq \Omega_{\vartheta,x}$, where $\Omega_{\vartheta,x}$ is the stabilizer of x in Ω_ϑ.*

Proof. Let $W_\sigma = N^\sigma/T^\vartheta$ be the subgroup of W whose elements can be represented by σ-fixed elements of N. If $n \in N^\sigma$, then $\vartheta(n) \equiv n$ modulo T. Hence W_σ is a subgroup of W^ϑ. Let $\widetilde{W}_{\vartheta,x}$ denote the stabilizer of x in $\widetilde{W}_\vartheta$. I claim that the projection $\pi\colon \widetilde{W}_\vartheta \to W^\vartheta$ sends $\widetilde{W}_{\vartheta,x}$ onto W_σ and gives an isomorphism

$$\widetilde{W}_{\vartheta,x} \xrightarrow{\sim} W_\sigma . \tag{3.14}$$

If $w \in W^\vartheta$ is the projection of an element of $\widetilde{W}_{\vartheta,x}$ then $w \cdot x - x \in Y_\vartheta$. By equation (3.3), there are $v \in V$ and $y \in Y$ such that

$$w \cdot x - x = (\vartheta - 1)\, v + y .$$

Setting $s = \exp(x)$, $t = \exp(v)$, we have

$$w(s) = t^{-1}\vartheta(t)s .$$

By [21, 8.2(4)] we may choose $\dot{w} \in N^\vartheta$ such that $w = \dot{w}T$. Then the element $n = t\dot{w}$ belongs to N^σ and $nT = w$. Thus, the projection (3.14) maps $\widetilde{W}_{\vartheta,x}$ into W_σ. The argument is reversible, showing that $\pi(\widetilde{W}_{\vartheta,x}) = W_\sigma$. Finally, since the kernel of π is torsion free and $\widetilde{W}_{\vartheta,x}$ is finite, the map (3.14) is injective, completing the proof of (3.14). With this in hand, the rest of the argument is entirely similar to that of Proposition 2.1, and is left to the reader.

REMARK. From Table 2, we see that for $\vartheta \neq 1$ the group G^σ has at most two components, is always connected in types ${}^2A_{2n}$, ${}^3D_4$ and ${}^2E_6$, and is disconnected in types ${}^2A_{2n-1}$ and ${}^2D_{n+1}$ exactly when the Kac coordinates $(s_\iota)_{\iota \in \widetilde{I}}$ are fixed by the nontrivial symmetry of $\mathcal{D}(\mathfrak{g}, \vartheta)$ (cf. [21, 9.8]).

3.7 The center of G^σ

Let $\sigma = \vartheta\exp(x)$, with x contained in the facet C^J_ϑ of $\overline{C}_\vartheta$. The center Z_σ of G^σ centralizes $T^\sigma = T^\vartheta$, hence is contained in T^ϑ. Since G^ϑ has trivial center, the character group $X^*(T^\vartheta)$ is generated by restrictions of roots of T. It follows that the character group of Z_σ is

$$X^*(Z_\sigma) = X^*(T^\vartheta)/\mathbf{Z}\widehat{\Phi}^J_\vartheta = \mathbf{Z}\widehat{\Delta}_\vartheta/\mathbf{Z}\widehat{\Delta}^J_\vartheta\,,$$

where $\widehat{\Delta}^J_\vartheta$ is the set of gradients of the affine roots β_ι for $\iota \in J$. Since all but at most one root of $\widehat{\Delta}^J_\vartheta$ are contained in $\widehat{\Delta}_\vartheta$, the possible exception being $-\widetilde{\beta}_0 = -\sum_{i\in I} b_i\beta_i$, it follows that $X^*(Z_\sigma)$ has rank equal to $|I| - |J|$ and the torsion subgroup of $X^*(Z_\sigma)$ is cyclic of order equal to the $\gcd\{b_\iota \,:\, \iota \in \widetilde{I} - J\}$. For example, Z_σ is connected if $0 \notin J$.

3.8 Isolated automorphisms

A semisimple automorphism $\sigma \in \mathrm{Aut}(\mathfrak{g})$ is *isolated* if the fixed-point subalgebra $\mathfrak{g}^\sigma$ is semisimple. Such a σ is necessarily torsion, lest the Zariski-closure of $\langle\sigma\rangle$ contain a nontrivial torus in the center of G^σ. The previous section shows that, for $x \in \overline{C}_\vartheta$, the automorphism $\sigma = \vartheta\exp(x)$ is isolated exactly when x is a vertex of C_ϑ. Hence every isolated automorphism of $\mathfrak{g}$ is conjugate to some

$$\sigma_\iota := \vartheta\exp(v_\iota)\,, \quad \iota \in \widetilde{I}\,,$$

where $v_\iota = \langle\widetilde{\gamma}_0, \mu_\iota\rangle^{-1}\check{\mu}_\iota$ are the vertices of C_ϑ (see Section 3.3). The order m_ι of σ_ι is given by

$$m_\iota = c_\iota f_\iota = f\, b_\iota\,.$$

From Section 3.7, the center of G^{σ_ι} is cyclic of order b_ι, generated by $\sigma_i^f = \exp(f v_\iota)$. Equivalently, the center of $\langle\sigma_\iota\rangle G^{\sigma_\iota}$ is generated by σ_ι.

4. The various cases

4.1 ${}^2A_{2n}$

Here $\mathfrak{g} = \mathfrak{sl}_{2n+1}$. Instead of writing V as a quotient, as we did in Example 1 of Section 2.2, it is convenient now to express V as a cross-section of that quotient:

$$V = \{(x_1, \ldots, x_{2n+1}) \in \mathbf{R}^{2n+1} \,:\, \sum_{i=1}^{2n+1} x_i = 0\}\,.$$

We have $\mathrm{Aut}(\mathfrak{g}) = \langle\vartheta\rangle \cdot PGL_{2n+1}$ with pinned automorphism ϑ of order two, acting on V by

$$\vartheta \cdot (x_1, \dots, x_{2n+1}) = (-x_{2n+1}, \dots, -x_1)\,.$$

Hence

$$V^{\vartheta} = \{(x_1, \dots, x_n, 0, -x_n, \dots, -x_1) \,:\, x_i \in \mathbf{R}\}$$

can be identified with $\mathbf{R}^n$ via the first n coordinates and we may take $I = \{1, 2, \dots, n\}$ as the indexing set for the ϑ-orbits on $\{1, 2, \dots, 2n\}$. The lattice Y_ϑ has basis $\{\check{\mu}_i \,:\, 1 \le i \le n\}$, where

$$\check{\mu}_i = \tfrac{1}{2}(e_1 + e_2 + \cdots + e_i)\,,$$

and $\{e_1, \dots, e_n\}$ is the standard basis of $\mathbf{R}^n$. The simple roots $\alpha_i = x_i - x_{i+1}$ on V restrict to V^ϑ as

$$\beta_i = \begin{cases} x_i - x_{i+1} & \text{for } 1 \le i < n\,, \\ x_n & \text{for } i = n\,. \end{cases}$$

For $1 \le i < n$ the ϑ-equivalence classes $a_i = \{\alpha_i, \alpha_{2n+1-i}\}$ have type I and $a_n = \{\alpha_n, \alpha_{n+1}, \alpha_n + \alpha_{n+1}\}$ has type II, so we have $f_i = 2$ for $1 \le i < n$, $f_n = 4$ and

$$\gamma_i = \begin{cases} 2\beta_i = 2(x_i - x_{i+1}) & \text{for } 1 \le i < n\,, \\ 4\beta_n = 4x_n & \text{for } i = n\,. \end{cases}$$

The root system Φ_ϑ, with basis $\Delta_\vartheta = \{\gamma_1, \dots, \gamma_n\}$, has type C_n. The highest root $\tilde{\gamma}_0$ is given by

$$\tilde{\gamma}_0 = 2\gamma_1 + 2\gamma_2 + \cdots + 2\gamma_{n-1} + \gamma_n = 4x_1$$

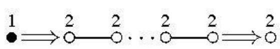
and arises from the type-II equivalence class $a_0 = \{\alpha_0, \vartheta\alpha_0, \alpha_0 + \vartheta\alpha_0\}$, where $\alpha_0 = \alpha_1 + \alpha_2 + \cdots + \alpha_n = x_1 - x_{n+1}$. We have $c_i = \langle\tilde{\gamma}_0, \check{\mu}_i\rangle = 2$ for all i. It follows that the alcove $C_\vartheta \subset V^\vartheta$ is defined by the inequalities

$$\tfrac{1}{4} > x_1 > x_2 > \cdots > x_n > 0$$

and has vertices $v_0 = 0$ and $v_i = \frac{1}{2}\check{\mu}_i$ for $1 \le i \le n$. For $1 \le i \le n$ we have $b_i = 2c_i/2 = 2$ and

$$\tilde{\beta}_0 = f^{-1}\tilde{\gamma}_0 = 2x_1$$

(which equals $2\beta_1$ if $n = 1$). Thus, we get the diagrams $\mathcal{D}(\mathfrak{g}, \vartheta)$ in Table 2:



if $n > 1$

and $\overset{1}{\bullet}\Rrightarrow\overset{2}{\circ}$ if $n=1$. The group Ω_ϑ is trivial, so every torsion element in ϑG is G-conjugate to a unique one of the form $\vartheta\exp(x)$ with $x \in \bar{C}_\vartheta \cap V_{\mathbf{Q}}$. The isolated automorphisms

$$\sigma_i := \vartheta\exp(v_i)$$

have order

$$m_i = f\,b_i = \begin{cases} 2 & \text{if } i = 0\,, \\ 4 & \text{if } 1 \leq i \leq n\,. \end{cases}$$

The fixed-point subalgebras are

$$\mathfrak{g}^{\sigma_i} \simeq \mathfrak{sp}_{2i} \oplus \mathfrak{so}_{2(n-i)+1}\,.$$

By Lemma 3.9 the fixed-point subgroups G^{σ_i} are connected. They have center of order two, for $1 \leq i \leq n$ with trivial center for $i = 0$. Indeed, we have

$$G^{\sigma_i} \simeq Sp_{2i} \times SO_{2(n-i)+1}\,.$$

To see this directly via linear algebra, let $(\cdot \mid \cdot)$ be the usual dot-product on $\mathbf{C}^{2n+1}$, let J be the matrix equal to one on the anti-diagonal and zero elsewhere, and let s_i be a diagonal matrix with characteristic polynomial $(t^2+1)^i(t-1)^{2(n-i)+1}$. Then the bilinear form

$$\langle u, v\rangle_i := (s_i u \mid Jv)$$

is orthogonal on the $1-$eigenspace of s_i and symplectic on the sum of the imaginary eigenspaces of s_i. The subgroup of GL_{2n+1} preserving $\langle\,.\,,.\rangle_i$ is $Sp_{2i} \times O_{2(n-i)+1}$, whose image in PGL_{2n+1} is isomorphic to $Sp_{2i} \times SO_{2(n-i)+1}$.

4.2 ${}^2A_{2n-1}$, $n \geq 2$

Here $\mathfrak{g} = \mathfrak{sl}_{2n}$ and

$$V = \{(x_1, \ldots, x_{2n}) \in \mathbf{R}^{2n} : \sum_{i=1}^{2n} x_i = 0\}\,.$$

We have $\mathrm{Aut}(\mathfrak{g}) = \langle\vartheta\rangle \cdot PGL_{2n}$ with pinned automorphism ϑ of order two, acting on V by

$$\vartheta \cdot (x_1, \ldots, x_{2n}) = (-x_{2n}, \ldots, -x_1)\,.$$

Hence

$$V^\vartheta = \{(x_1, \ldots, x_n, -x_n, \ldots, -x_1) : x_i \in \mathbf{R}\}$$

can be identified with $\mathbf{R}^n$ via the first n coordinates and we may take $I = \{1, 2, \ldots, n\}$ as the indexing set for the ϑ-orbits on $\{1, 2, \ldots, 2n\}$. The lattice Y_ϑ has basis $\{\check{\mu}_i : 1 \leq i \leq n\}$, where

$$\check{\mu}_i = \tfrac{1}{2}(e_1 + e_2 + \cdots + e_i)$$

and $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbf{R}^n$. The simple roots $\alpha_i = x_i - x_{i+1}$ on V restrict to V^ϑ as

$$\beta_i = \begin{cases} x_i - x_{i+1} & \text{for } 1 \leq i < n\,, \\ 2x_n & \text{for } i = n\,. \end{cases}$$

All ϑ-equivalence classes have type I and are ϑ-orbits on the roots. We have

$$\gamma_i = \begin{cases} 2\beta_i = 2(x_i - x_{i+1}) & \text{for } 1 \leq i < n\,, \\ \beta_n = 2x_n & \text{for } i = n\,. \end{cases}$$

The root system Φ_ϑ, with basis $\Delta_\vartheta = \{\gamma_1, \ldots, \gamma_n\}$, has type B_n and the highest root $\widetilde{\gamma}_0$ is given by

$$\widetilde{\gamma}_0 = \sum_{i=1}^{n} c_i\gamma_i = \gamma_1 + 2\gamma_2 + \cdots + 2\gamma_{n-1} + 2\gamma_n = 2(x_1 + x_2)\,,$$

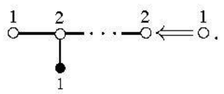
arising from the ϑ-orbit $a_0 = \{x_1 - x_{2n-1}, x_2 - x_{2n}\}$. It follows that the alcove $C_\vartheta \subset V^\vartheta$ is defined by the inequalities

$$\tfrac{1}{2} - x_2 > x_1 > x_2 > \cdots > x_n > 0$$

and has vertices

$$v_0 = 0\,, \quad v_1 = \check{\mu}_1\,, \quad v_i = \tfrac{1}{2}\check{\mu}_i \quad \text{for} \quad 2 \leq i \leq n\,.$$

We have $\widetilde{\beta}_0 = f^{-1}\widetilde{\gamma}_0 = x_1 + x_2$, so we get the diagram $\mathcal{D}(\mathfrak{g}, \vartheta)$ in Table 2:



The group Ω_ϑ has order two, and the nontrivial element $\rho \in \Omega_\vartheta$ acts on V^ϑ by

$$\rho \cdot (x_1, x_2, \ldots, x_{n-1}, x_n) = (\tfrac{1}{2} - x_1, x_2, \ldots, x_{n-1}, x_n)\,.$$

Hence $\rho \cdot v_0 = v_1$ and ρ gives the nontrivial symmetry of the diagram $\mathcal{D}(\mathfrak{g}, \vartheta)$. For $1 \leq i \leq n$, the isolated automorphism

$$\sigma_i := \vartheta \exp(v_i)$$

has order

$$m_i = f\,b_i = \begin{cases} 2 & \text{if } i = 1 \text{ or } n\,, \\ 4 & \text{if } 1 < i < n\,. \end{cases}$$

We will ignore $i = 1$, since σ_1 is conjugate to $\sigma_0 = \vartheta$. For $0 \le i \le n,\ i \ne 1$, the fixed-point subalgebra is

$$\mathfrak{g}^{\sigma_i} \simeq \mathfrak{so}_{2i} \ominus \mathfrak{sp}_{2(n-i)}\,.$$

For $i = 0$ the fixed-point subgroup $G^{\vartheta} = Sp_{2n}/\{\pm I\}$ is connected with trivial center. For $1 < i \le n$ the fixed-point group G^{σ_i} has two components and has center of order two. Indeed, we have

$$G^{\sigma_i} \simeq [O_{2i} \times Sp_{2(n-i)}]/\{\pm I_{2n}\}\,.$$

To see this directly via linear algebra, let $(\cdot \mid \cdot)$ be the usual dot-product on $\mathbf{C}^{2n}$, let J be the matrix equal to one on the anti-diagonal and zero elsewhere, and let s_i be a diagonal matrix with characteristic polynomial $(t^2+1)^{(n-i)}(t-1)^{2i}$. Then the bilinear form

$$\langle u, v\rangle_i := (s_i u \mid Jv)$$

is orthogonal on the $1-$eigenspace of s_i and symplectic on the sum of the imaginary eigenspaces of s_i. The subgroup of GL_{2n} preserving $\langle .\,,.\rangle_i$ is $O_{2i} \times Sp_{2(n-i)}$, which has kernel $\{\pm I_{2n}\}$ when projected into PGL_{2n}.

4.3 ${}^2D_{n+1}$

Here $\mathfrak{g} = \mathfrak{so}_{2n+2}$ and $V = \mathbf{R}^{n+1}$. We have

$$\mathrm{Aut}(\mathfrak{g}) = \langle\vartheta\rangle \cdot PSO_{2n+2} = O_{2n+2}/\{\pm I\}$$

with pinned automorphism ϑ of order two, acting on V by

$$\vartheta \cdot (x_1, \ldots, x_n, x_{n+1}) = (x_1, \ldots, x_n, -x_{n+1})\,.$$

Hence

$$V^{\vartheta} = \{(x_1, \ldots, x_n, 0) \,:\, x_i \in \mathbf{R}\}$$

can be identified with $\mathbf{R}^n$ via the first n coordinates and we may take $I = \{1, 2, \ldots, n\}$ as the indexing set for the ϑ-orbits on $\{1, 2, \ldots, n+1\}$. The lattice Y_{ϑ} has basis $\{\check{\mu}_i \,:\, 1 \le i \le n\}$, where

$$\check{\mu}_i = \begin{cases} e_1 + \cdots + e_i & \text{if } 1 \le i < n\,, \\ \frac{1}{2}(e_1 + \cdots + e_n) & \text{if } i = n\,. \end{cases}$$

The simple roots

$$\alpha_i = x_i - x_{i+1} \ (1 \le i \le n)\,, \qquad \alpha_{n+1} = x_n + x_{n+1}$$

on V restrict to V^ϑ as

$$\beta_i = \begin{cases} x_i - x_{i+1} & \text{for } 1 \le i < n\,, \\ x_n & \text{for } i = n\,. \end{cases}$$

All ϑ-equivalence classes have type I, and are ϑ-orbits on the roots. We have

$$\gamma_i = \begin{cases} \beta_i = x_i - x_{i+1} & \text{for } 1 \le i < n\,, \\ 2\beta_n = 2x_n & \text{for } i = n\,. \end{cases}$$

The root system Φ_ϑ, with basis $\Delta_\vartheta = \{\gamma_1, \dots, \gamma_n\}$, has type C_n and the highest root $\widetilde{\gamma}_0$ is given by

$$\widetilde{\gamma}_0 = \sum_{i=1}^{n} c_i\gamma_i = 2\gamma_1 + 2\gamma_2 + \cdots + 2\gamma_{n-1} + \gamma_n = 2x_1\,,$$

arising from the ϑ-orbit $a_0 = \{x_1 - x_{n+1}, x_1 + x_{n+1}\}$. It follows that the alcove $C_\vartheta \subset V^\vartheta$ is defined by the inequalities

$$\tfrac{1}{2} > x_1 > x_2 > \cdots > x_n > 0$$

and has vertices

$$v_0 = 0, \quad v_i = \tfrac{1}{2}(e_1 + \cdots + e_i) \quad \text{for} \quad 1 \le i \le n\,.$$

We have $\widetilde{\beta}_0 = f^{-1}\widetilde{\gamma}_0 = x_1$, so we get the diagram $\mathcal{D}(\mathfrak{g}, \vartheta)$ in Table 2:

$$\overset{1}{\bullet}\Leftarrow\overset{1}{\circ}-\overset{1}{\circ}-\cdots-\overset{1}{\circ}\Rightarrow\overset{1}{\circ}.$$

The group Ω_ϑ has order two, and the nontrivial element $\rho \in \Omega_\vartheta$ acts on V^ϑ by

$$\rho \cdot (x_1, x_2, \dots, x_{n-1}, x_n) = (\tfrac{1}{2} - x_n, \tfrac{1}{2} - x_{n-1}, \dots, \tfrac{1}{2} - x_2, \tfrac{1}{2} - x_1)\,.$$

Hence $\rho \cdot v_i = v_{n-i}$ and ρ gives the nontrivial symmetry of the diagram $\mathcal{D}(\mathfrak{g}, \vartheta)$. For $1 \le i \le n$, the isolated automorphism

$$\sigma_i := \vartheta \exp(v_i)$$

has order

$$m_i = f\, b_i = 2 \quad \text{for } 1 \le i \le n\,.$$

The fixed-point subalgebra is

$$\mathfrak{g}^{\sigma_i} \simeq \mathfrak{so}_{2(n-i)+1} \oplus \mathfrak{so}_{2i+1} .$$

Since all $b_i = 1$, the fixed-point subgroup G^{σ_i} has trivial center for all i and is connected unless n is even and $i = n/2$. In that case, there are two components. More precisely, for $i \neq n/2$ we have

$$G^{\sigma_i} \simeq SO_{2(n-i)+1} \times SO_{2i+1} ,$$

and for $n = 2k$ we have

$$G^{\sigma_k} \simeq 2 \cdot [SO_{2k+1} \times SO_{2k+1}] ,$$

where the outer involution switches the two components.

To see this directly via linear algebra, note that the automorphism σ_i is conjugation by an element of order two in O_{2n+2} having characteristic polynomial $(t+1)^{2(n-i)+1}(t-1)^{2i+1}$.

4.4 $^3D_4$

Here $\mathfrak{g} = \mathfrak{so}_8$ has $\mathrm{Aut}(\mathfrak{g}) = S_3 \cdot PSO_8$ and we take $\vartheta \in S_3$ of order three. Denote the set of simple roots of D_4 by $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, where α_2 corresponds to the branch node, and let $\check{\omega}_i$ be the fundamental co-weight dual to α_i. We write $\widetilde{I} = \{0, 1, 2\}$, where "1" and "2" stand for the ϑ-orbits $\{1, 3, 4\}$ and $\{2\}$, respectively. The equivalence classes a and corresponding restricted roots $\beta_a \in \widehat{\Phi}_\vartheta$ and roots $\gamma_a \in \Phi_\vartheta$ are as follows:

a	β_a	γ_a
α_1 α_3 α_4	β_1	$\gamma_1 = 3\beta_1$
α_2	β_2	$\gamma_2 = \beta_2$
$\alpha_2 + \alpha_1$ $\alpha_2 + \alpha_3$ $\alpha_2 + \alpha_4$	$\beta_1 + \beta_2$	$\gamma_1 + 3\gamma_2 = 3(\beta_1 + \beta_2)$
$\alpha_2 + \alpha_3 + \alpha_4$ $\alpha_2 + \alpha_4 + \alpha_1$ $\alpha_2 + \alpha_1 + \alpha_3$	$2\beta_1 + \beta_2 = \widetilde{\beta}_0$	$2\gamma_1 + 3\gamma_2 = \widetilde{\gamma}_0 = 3(2\beta_1 + \beta_2)$
$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$	$3\beta_1 + \beta_2$	$\gamma_1 + \gamma_2 = 3\beta_1 + \beta_2$
$\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$	$3\beta_1 + 2\beta_2$	$\gamma_1 + 2\gamma_2 = 3\beta_1 + 2\beta_2$

From Table 2 we have the Kac diagram $\mathcal{D}(\mathfrak{g},\vartheta)$, with label b_ι above the node $\textcircled{\iota}$:

$$\overset{1}{\textcircled{0}}\!-\!\!\!-\!\overset{2}{\textcircled{1}} \Leftarrow \overset{1}{\textcircled{2}}$$

which shows that the isolated automorphisms $\sigma_\iota = \vartheta\exp(v_\iota)$, of order $m_\iota = c_\iota f_\iota = f\,b_\iota$, where v_ι are the vertices of C_ϑ, have semisimple fixed-point groups G^{σ_ι} of types G_2, $A_1\times A_1$, A_2, for $\iota = 0,1,2$ respectively. More precise information, including the exact isomorphism type of G^{σ_ι}, is given in the next table.

ι	f_ι	c_ι	b_ι	$m_\iota = \lvert\sigma_\iota\rvert$	$\check{\mu}_\iota$	v_ι	G^{σ_ι}
0	3	1	1	3	$0 \in V^\vartheta$	$0 \in V^\vartheta$	G_2
1	3	2	2	6	$\frac{1}{3}(\check{\omega}_1+\check{\omega}_3+\check{\omega}_4)$	$\frac{1}{6}(\check{\omega}_1+\check{\omega}_3+\check{\omega}_4)$	SO_4
2	1	3	1	3	$\check{\omega}_2$	$\frac{1}{3}\check{\omega}_2$	PGL_3

Since $\Omega_\vartheta = 1$, Lemma 3.9 shows that each G^{σ_ι} is connected. From Section 3.7, the center of G^{σ_ι} is trivial for $\iota = 0,2$. This gives $G^\vartheta \simeq G_2$ and $G^{\sigma_2} \simeq PGL_3$. Since 3.7 also shows that the center of G^{σ_1} has order two, we can pin down the isomorphism type of G^{σ_1} as follows. Its simply-connected cover $G^{\sigma_1}_{sc} \simeq SL_2\times SL_2$. The weight β_1 appears in $\mathfrak{g}$ and

$$\langle\beta_1,\beta_0\rangle = -\langle\beta_1,\beta_2\rangle = 1\,.$$

Hence the center of each SL_2 factor is nontrivial on $\mathfrak{g}$, so the kernel of the covering $G^{\sigma_1}_{sc}\to G^{\sigma_1}$ must be the diagonal embedding $\Delta\mu_2$ of $\mu_2 = \{\pm 1\}$ into the center of $G^{\sigma_1}_{sc}$. Thus, we find that $G^{\sigma_1}\simeq SO_4$.

With more work, one can also see this by decomposing $\mathfrak{g} = \mathfrak{so}_8$ under G^{σ_1}. Let Sym^m be the irreducible representation of SL_2 on the m^{th} symmetric power of $\mathbf{C}^2$ and write $\mathrm{Sym}^{m,n} := \mathrm{Sym}^m\otimes\mathrm{Sym}^n$ for the irreducible representations of $SL_2\times SL_2$. For each ϑ-orbit a we compute the polynomial

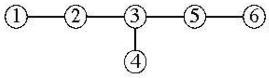
$$\det(t-\sigma_1|\mathfrak{g}_a) = t^{|a|} - e^{2\pi i\langle\gamma_a,v_1\rangle}\,,$$

as in (3.9). This leads to the decomposition of the representation of G^{σ_1} on the σ_1-eigenspace $\mathfrak{g}(\zeta)$ for each sixth root of unity ζ, as follows:

$$\begin{aligned} &\mathfrak{g}(1)\simeq \mathrm{Sym}^{2,0}\oplus\mathrm{Sym}^{0,2}, && \mathfrak{g}(-1)\simeq\mathrm{Sym}^{3,1},\\ &\mathfrak{g}(e^{2\pi i/3})\simeq\mathfrak{g}(e^{4\pi i/3})\simeq\mathrm{Sym}^{0,2}, && \mathfrak{g}(e^{\pi i/3})\simeq\mathfrak{g}(e^{5\pi i/3})\simeq\mathrm{Sym}^{1,1}\,. \end{aligned}\tag{4.1}$$

The parity of m,n for the various $\mathrm{Sym}^{m,n}$ appearing in $\mathfrak{g}$ and the fact that G^{σ_1} is faithful on $\mathfrak{g}$, confirm that $G^{\sigma_1}\simeq SO(4)$.

4.5 EXAMPLE: $^2E_6$

We label the E_6 Dynkin graph as:  , so that the ϑ-orbits of simple roots are

$$a_1 = \{\alpha_1, \alpha_6\}, \quad a_2 = \{\alpha_2, \alpha_5\}, \quad a_3 = \{\alpha_3\}, \quad a_4 = \{\alpha_4\},$$

and

$$\gamma_1 = 2\beta_1, \quad \gamma_2 = 2\beta_2, \quad \gamma_3 = \beta_3, \quad \gamma_4 = \beta_4.$$

The highest root of Φ_ϑ is $\tilde{\gamma}_0 = \gamma_{a_0}$, where

$$a_0 = \{\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \ \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 + 2\alpha_5 + \alpha_6\},$$

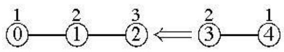
so that

$$\tilde{\gamma}_0 = 2\gamma_1 + 3\gamma_2 + 4\gamma_3 + 2\gamma_4,$$

and therefore

$$\tilde{\beta}_0 = 2\beta_1 + 3\beta_2 + 2\beta_3 + \beta_4,$$

giving the Kac diagram from Table 2, with label b_ι above the node ⓘ:

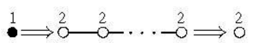
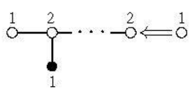
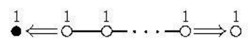
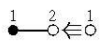
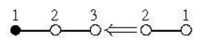
 .

Since ϑ acts by inversion on $\Omega \simeq \mathbf{Z}/3$, we have $\Omega_\vartheta = 1$, so Lemma 3.9 shows that each G^{σ_ι} is connected for all ι. From Section 3.7, the center of G^{σ_ι} is trivial for $\iota = 0, 4$. This gives $G^\vartheta \simeq F_4$ and $G^{\sigma_4} \simeq PSp_8$. The remaining centers have orders $b_\iota = 2, 3, 2$, for $\iota = 1, 2, 3$ respectively. We can pin down the isomorphism types as we did for $^3D_4$, by computing $\langle \beta_i, \check{\beta}_{i\pm 1}\rangle$, to arrive at the table below, where $\Delta\mu_n$ denotes a diagonal embedding of the group of n^{th} roots of unity into the center of a product of simply connected groups.

ι	f_ι	c_ι	b_ι	$m_\iota = \lvert\sigma_\iota\rvert$	$\check{\mu}_\iota$	v_ι	G^{σ_ι}
0	1	1	1	2	$0 \in V^\vartheta$	$0 \in V^\vartheta$	F_4
1	2	2	2	4	$\frac{1}{2}(\check{\omega}_1 + \check{\omega}_6)$	$\frac{1}{4}(\check{\omega}_1 + \check{\omega}_6)$	$[SL_2 \times Sp_6]/\Delta\mu_2$
2	2	3	3	6	$\frac{1}{2}(\check{\omega}_2 + \check{\omega}_5)$	$\frac{1}{6}(\check{\omega}_2 + \check{\omega}_5)$	$[SL_3 \times SL_3]/\Delta\mu_3$
3	1	4	2	4	$\check{\omega}_3$	$\frac{1}{4}\check{\omega}_3$	$[SL_4 \times SL_2]/\Delta\mu_2$
4	1	2	1	2	$\check{\omega}_4$	$\frac{1}{2}\check{\omega}_4$	PSp_8

5. Twisted Coxeter elements

We close with a twisted analogue of Kostant's result on principal elements, mentioned in Section 2.5. Let $w \in W$ be the product, taken in any order, of a set of representatives for the ϑ-orbits on the set $\{r_i : 1 \leq i \leq \ell\}$ of simple reflections in W. The element $\vartheta w \in \vartheta W$ is called a *ϑ-twisted Coxeter element* [20]. Such elements form a single W-conjugacy class in ϑW, independent of the choice of representatives or the order in the product. The order h_ϑ of ϑw is the *ϑ-twisted Coxeter number*. By construction, the length $\ell(w) = \ell_\vartheta$ is the rank of G^ϑ. These are tabulated below, along with the sum $\mathrm{ht}(\beta_0)$ of the labels of the diagrams $\mathcal{D}(\mathfrak{g}, \vartheta)$, and the degrees of the basic W-invariant polynomials affording a primitive f^{th} root of unity as ϑ-eigenvalue.

Type	$\mathcal{D}(\mathfrak{g}, \vartheta)$	$\widehat{\Phi}_\vartheta$	h_ϑ	$\mathrm{ht}(\beta_0)$	f-degrees
${}^2A_2$		B_1	6	3	3
${}^2A_{2n}$ $(n \geq 2)$		B_n	$4n+2$	$2n+1$	$3, 5, \ldots, 2n+1$
${}^2A_{2n-1}$ $(n \geq 3)$		C_n	$4n-2$	$2n-1$	$3, 5, \ldots, 2n-1$
${}^2D_{n+1}$ $(n \geq 2)$		B_n	$2n+2$	$n+1$	$n+1$
${}^3D_4$		$\widehat{G}_2$	12	4	4, 4
${}^2E_6$		$\widehat{F}_4$	18	9	5, 9

B. Gross pointed out to me that $h_\vartheta = f \cdot \mathrm{ht}(\beta_0)$, meaning that a torsion automorphism $\sigma \in \vartheta G$ with Kac coordinates $s_\iota = 1$ for all $\iota \in \widetilde{I}$ has order h_ϑ. In fact, the table shows that twisted Coxeter numbers have the properties:

$$h_\vartheta = f \cdot \mathrm{ht}(\beta_0) = \frac{|\Phi|}{\ell_\vartheta} = f \cdot \text{largest } f\text{-degree}$$

generalizing other well-known properties of ordinary Coxeter numbers. This indicates that σ might be a lift to $\mathrm{Aut}(G)$ of a twisted Coxeter element. We will prove that this is the case:

PROPOSITION 5.1. *Let $\sigma \in \vartheta G$ be a torsion automorphism with Kac coordinates $s_\iota = 1$ for all $\iota \in \tilde{I}$. Then σ preserves a Cartan subalgebra of $\mathfrak{g}$ and acts there via a ϑ-twisted Coxeter element.*

For $\vartheta = 1$ this is Kostant's result, proved in [14], and mentioned in Section 2.5 above. We will use some of Kostant's arguments in what follows, but instead of his theory of cyclic elements, we will invoke the classification of torsion automorphisms. The main point is the following lemma, which is also used in [9]:

LEMMA 5.2. *Let $\sigma \in \vartheta N$ be a torsion automorphism of $\mathfrak{g}$ of order m, let L denote the number of σ-orbits on the set Φ of roots of T in $\mathfrak{g}$. Then*

$$\dim \mathfrak{t}^\vartheta \leq \dim \mathfrak{t}^\sigma + L \tag{5.1}$$

and equality implies the following:

1. *$\mathfrak{g}^\sigma$ is abelian and $\mathfrak{t}^\sigma = 0$, so that $\dim \mathfrak{t}^\vartheta = L$;*
2. *the projection of σ to ϑW has the same order m as σ;*
3. *$m \geq h_\vartheta$, with equality if and only if σ has all Kac coordinates $s_\iota = 1$.*

Proof. Partition $\Phi = \Phi_1 \cup \cdots \cup \Phi_L$ into σ-orbits of size $n_i = |\Phi_i|$ and let $\mathfrak{g}_i$ be the span of the root vectors X_α for $\alpha \in \Phi_i$. Then

$$\mathfrak{g}^\sigma = \mathfrak{t}^\sigma + \sum_{i=1}^{L} \mathfrak{g}_i^\sigma .$$

Since σ^{n_i} fixes every root in Φ_i, it acts on $\mathfrak{g}_i$ as scalar multiplication by some $z_i \in \mathbf{C}^\times$ and we have

$$\dim \mathfrak{g}_i^\sigma = \begin{cases} 1 & \text{if } z_i = 1 , \\ 0 & \text{if } z_i \neq 1 . \end{cases}$$

On the other hand, since $\sigma \in \vartheta G$, the subalgebra $\mathfrak{t}^\vartheta$ is G-conjugate to a Cartan subalgebra of $\mathfrak{g}^\sigma$. It follows that

$$\dim \mathfrak{t}^\vartheta \leq \dim \mathfrak{g}^\sigma = \dim \mathfrak{t}^\sigma + |\{i \,:\, z_i = 1\}| \leq \dim \mathfrak{t}^\sigma + L .$$

If equality holds at both steps, then $\mathfrak{t}^\vartheta$ and $\mathfrak{g}^\sigma$ are G-conjugate and $z_i = 1$ for all $1 \leq i \leq L$. Hence $\mathfrak{g}^\sigma$ is abelian and

$$\mathfrak{g}^\sigma = \mathfrak{t}^\sigma + \sum_{i=1}^{L} \mathbf{C} X_i ,$$

where X_i is a nonzero vector in $\mathfrak{g}_i^\sigma$. If $H \in \mathfrak{t}^\sigma$ then the value $\eta_i = \langle\alpha, H\rangle$ is constant for $\alpha \in \Phi_i$, and $[H, X_i] = \eta_i X_i$. But since $\mathfrak{g}^\sigma$ is abelian, we have all $\eta_i = 0$, so $\langle\alpha, H\rangle = 0$ for all $\alpha \in \Phi$, meaning that $H = 0$. Hence $\mathfrak{t}^\sigma = 0$ and assertion 1 holds. Moreover, since $\mathfrak{g}^\sigma$ is abelian it has empty root-system, so the Kac coordinates s_ι of σ are all non-zero and the order m of σ satisfies the inequality

$$m = f \cdot \sum_{\iota \in \tilde{I}} b_\iota s_\iota \geq f \cdot \mathrm{ht}(\beta_0) = h_\vartheta ,$$

with equality if and only if all $s_\iota = 1$. Assertion 3 is proved. The projection of σ to ϑW has order equal to the least common multiple n of $\{n_1, \ldots, n_L\}$ and $\sigma^n = I$ on $\mathfrak{t}$. If $z_i = 1$ for all i, then $\sigma^n = I$ on $\mathfrak{g}_i$ for all i, so $n = m$, completing the proof of the lemma.

Next, following Kostant, we have an inequality in the reverse direction. Assume now that $\mathfrak{t}^\sigma = 0$. Let $N_\sigma = \{\alpha \in \Phi^+ : \sigma\alpha \in -\Phi^+\}$. Then $|N_\sigma| = \ell(w)$, where ϑw is the projection of σ to ϑW, and $\ell(w)$ is the Coxeter length of w with respect to the base Δ. For each i, the intersection $\Phi_i \cap N_\sigma$ is nonempty. For otherwise, all roots in Φ_i would have the same sign, so their sum would be non-zero and σ-invariant, contradicting our assumption that $\mathfrak{t}^\sigma = 0$. Therefore, we have

$$\ell(w) = \sum_{i=1}^{L} |\Phi_i \cap N_\sigma| \geq L , \tag{5.2}$$

with equality if and only if $|\Phi_i \cap N_\sigma| = 1$ for all i.

We now prove Proposition 5.1, by computing the Kac coordinates of a lift $\sigma \in \vartheta N$ of a twisted Coxeter element ϑw in ϑW. From [20, 7.4(i)] we have that $\mathfrak{t}^\sigma = 0$. By the construction of w, we have $\ell(w) = \dim \mathfrak{t}^\vartheta$. From (5.2) we have $\dim \mathfrak{t}^\vartheta \geq L$. Hence we have equality in Lemma 5.2, so σ and ϑw have the same order, namely h_ϑ, and $s_\iota = 1$ for all ι. Since there is a unique torsion class in ϑG with these Kac coordinates, this proves Proposition 5.1.

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