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Objekttyp: **Article**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **55 (2009)**

Heft 1-2

PDF erstellt am: **19.09.2024**

Persistenter Link: https://doi.org/10.5169/seals-110099

L'Enseignement Mathématique (2) 55 (2009), 139–156

TERNARY CUBIC FORMS AND ÉTALE ALGEBRAS

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The configuration of inflection points on a nonsingular cubic curve in the complex projective plane has a well-known remarkable feature: a line through any two of the nine inflection points passes through a third inflection point. Therefore the inflection points and the 12 lines through them form a tactical configuration $(9_4, 12_3)$, which is the configuration of points and lines of the affine plane over the field with 3 elements ([3, p. 295], [7, p. 242]). This property was used by Hesse to show that the inflection points of a ternary cubic over the rationals are defined over a solvable extension, see [11, §110]. As a result, any ternary cubic can be brought to a normal form $x_1^3 + x_2^3 + x_3^3 - 3\lambda x_1 x_2 x_3$ over a solvable extension of the base field [1]). The purpose of this paper is to investigate this extension.

Throughout the paper, we denote by F an arbitrary field of characteristic different from 3, by F_s a separable closure of F and by $\Gamma = \mathrm{Gal}(F_s/F)$ its Galois group. Let V be a 3-dimensional F-vector space and let $f \in \mathsf{S}^3(V^*)$ be a cubic form on V. Assume that f has no singular zero in the projective plane $\mathbf{P}_V(F_s)$. Then the set $\mathfrak{I}(f) \subseteq \mathbf{P}_V(F_s)$ of inflection points has 9 elements. There are 12 lines in $\mathbf{P}_V(F_s)$ that contain three points of $\mathfrak{I}(f)$; they are called *inflectional lines*. Their set $\mathfrak{L}(f)$ is partitioned into four 3-element subsets $\mathfrak{T}_0$, $\mathfrak{T}_1$, $\mathfrak{T}_2$, $\mathfrak{T}_3$ called *inflectional triangles*, which have the property that each inflection point is incident to exactly one line of each triangle. Let $\mathfrak{T}(f) = \{\mathfrak{T}_0, \mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3\}$. There is a canonical map $\mathfrak{L}(f) \to \mathfrak{T}(f)$, which carries every inflectional line to the unique triangle that contains it. The Galois

*) The second author is partially supported by the Fund for Scientific Research F.R.S.–FNRS (Belgium).

[1]) We are grateful to the erudite anonymous referee who pointed out that the normal form of cubics was obtained by Hesse in [5, §20, Aufgabe 2] *before* he proved (in [6]) that the equation of inflection points is solvable by radicals.

group Γ acts on $\mathfrak{I}(f)$, hence also on $\mathfrak{L}(f)$ and $\mathfrak{T}(f)$, and the canonical map $\mathfrak{L}(f) \to \mathfrak{T}(f)$ is a triple covering of Γ-sets, in the terminology of [9, §2.2]. Galois theory associates to the Γ-set $\mathfrak{L}(f)$ a 12-dimensional étale F-algebra $L(f)$, which is a cubic étale extension of the 4-dimensional étale F-algebra $T(f)$ associated to $\mathfrak{T}(f)$. We show in §4 that if one of the inflectional triangles, say $\mathfrak{T}_0$, is defined over F, hence preserved under the Γ-action, then there are decompositions

$$T(f) \simeq F \times N, \qquad L(f) \simeq K \times M,$$

where N and K are cubic étale F-algebras whose corresponding Γ-sets are $\mathfrak{X}(N) = \{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3\}$ and $\mathfrak{X}(K) = \mathfrak{T}_0$ respectively, and M is a 9-dimensional étale F-algebra containing N, associated to K and a unit $a \in K^\times$. One can then identify the vector space V with K in such a way that

$$f(X) = \mathsf{T}_K(a^{-1}X^3) - 3\lambda\, \mathsf{N}_K(X) \quad \text{for some } \lambda \in F, \tag{0.1}$$

where T_K and N_K are the trace and the norm of the F-algebra K. Conversely, if f can be reduced to the form (0.1), then one of the inflectional triangles is defined over F, and $\mathfrak{X}(K)$ is isomorphic to the set of lines of the triangle. Note that the *(generalized) Hesse normal form*

$$a_1x_1^3 + a_2x_2^3 + a_3x_3^3 - 3\lambda x_1x_2x_3$$

is the particular case of (0.1) where $K = F \times F \times F$ (i.e., the Γ-action on $\mathfrak{X}(K)$ is trivial) and $a = (a_1^{-1}, a_2^{-1}, a_3^{-1})$. As an application, we show that the form $\mathsf{T}_K(X^3)$ can be reduced over F to a generalized Hesse normal form if and only if K has the form $F[\sqrt[3]{d}]$ for some $d \in F^\times$, see Example 4.4.

The 9-dimensional étale F-algebra M associated to a cubic étale F-algebra K and a unit $a \in K^\times$ was first defined by Markus Rost in relation with Morley's theorem. We are grateful to Markus for allowing us to quote from his private notes [10] in §2.

For background information on cubic curves, we refer to [3], Chapter 11 of [7], or [2].

1. Étale algebras over a field

An *étale F-algebra* is a finite-dimensional commutative F-algebra A such that $A \otimes_F F_s \simeq F_s \times \cdots \times F_s$; see [1, Ch. 5, §6] or [8, §18] for various other characterizations of étale F-algebras. For any étale F-algebra A, we denote by $\mathfrak{X}(A)$ the set of F-algebra homomorphisms $A \to F_s$. This is a finite set with cardinality $|\mathfrak{X}(A)| = \dim_F A$. Composition with automorphisms of F_s endows $\mathfrak{X}(A)$ with a Γ-set structure, and $\mathfrak{X}$ is a contravariant functor that defines an anti-equivalence of categories between the category Et_F of étale F-algebras and the category Set_Γ of finite Γ-sets, see [1, Ch. 5, § 10] or [8, (18.4)].

Let G be a finite group of automorphisms of an étale F-algebra A. The group G acts faithfully on the Γ-set $\mathfrak{X}(A)$.

PROPOSITION 1.1. *If G acts freely (i.e., without fixed points) on $\mathfrak{X}(A)$, then*

$$H^1(G, A^\times) = 1 .$$

Proof. The G-action on $\mathfrak{X}(A)$ maps each Γ-orbit on a Γ-orbit, since the actions of G and Γ commute. We may thus decompose $\mathfrak{X}(A)$ into a disjoint union

$$\mathfrak{X}(A) = \mathfrak{X}_1 \coprod \dots \coprod \mathfrak{X}_n ,$$

where each $\mathfrak{X}_i$ is a union of Γ-orbits permuted by G. Using the anti-equivalence between Et_F and Set_Γ, we obtain a corresponding decomposition of A into a direct product of étale F-algebras

$$A = A_1 \times \cdots \times A_n .$$

The G-action preserves each A_i, hence

$$H^1(G, A^\times) = H^1(G, A_1^\times) \times \cdots \times H^1(G, A_n^\times) .$$

It therefore suffices to prove that $H^1(G, A^\times) = 1$ when G acts transitively on the Γ-orbits in $\mathfrak{X}(A)$. These Γ-orbits are in one-to-one correspondence with the primitive idempotents of A. Let e be one of these idempotents and let $H \subseteq G$ be the subgroup of automorphisms that leave e fixed. Let also $B = eA$. The map $g \otimes b \mapsto g(b)$ for $g \in G$ and $b \in B$ induces isomorphisms of G-modules

$$A = \mathbf{Z}[G] \otimes_{\mathbf{Z}[H]} B , \qquad A^\times = \mathbf{Z}[G] \otimes_{\mathbf{Z}[H]} B^\times ,$$

hence the Eckmann–Faddeev–Shapiro lemma (see for instance [4, Prop. (6.2), p. 73]) yields an isomorphism

$$H^1(G, A^\times) \simeq H^1(H, B^\times) .$$

Now, B is a field and each element $h \in H$ restricts to an automorphism of B. Let $\xi \in \mathfrak{X}(A)$ be such that $\xi(e) = 1$, hence $\xi(x) = \xi(ex)$ for all $x \in A$. If $h \in H$ restricts to the identity on B then

$$e\, h(x) = h(ex) = ex \qquad \text{for all } x \in A ,$$

and hence

$$\xi\big(h(x)\big) = \xi(x) \quad \text{for all } x \in A.$$

It follows that h leaves ξ fixed, hence $h = 1$ since G acts freely on $\mathfrak{X}(A)$. Therefore H embeds injectively in the group of automorphisms of B. Hilbert's Theorem 90 then yields $H^1(H, B^\times) = 1$, see [8, (29.2)].

2. Morley algebras

Let K be an étale F-algebra of dimension 3. To every unit $a \in K^\times$ we associate an étale F-algebra $M(K, a)$ of dimension 9 by a construction due to Markus Rost [10], which will be crucial for the description of the Γ-action on inflectional lines of a nonsingular cubic, see Theorem 3.2.

Definition 2.1. Let D be the *discriminant algebra* of K (see [8, p. 291]); this is a 2-dimensional étale F-algebra such that $K \otimes_F D$ is the S_3-Galois closure of K, see [8, § 18.C]. We thus have F-algebra automorphisms σ, ρ of $K \otimes_F D$ such that

$$\sigma|_D = \mathrm{Id}_D\,, \quad \rho|_K = \mathrm{Id}_K\,, \quad \sigma^3 = \rho^2 = \mathrm{Id}_{K\otimes D}\,, \qquad \text{and} \qquad \rho\sigma = \sigma^2\rho\,.$$

We identify each element $x \in K$ with its image $x \otimes 1$ in $K \otimes_F D$ and denote its norm by $\mathsf{N}_K(x)$.

Now, fix an element $a \in K^\times$. Let s, t be indeterminates and consider the quotient F-algebra

$$A = K \otimes_F D[s,t]\big/\big(s^3 - \sigma^2(a)\,\sigma(a)^{-1},\ t^3 - \mathsf{N}_K(a)\big)\,.$$

Since the characteristic is different from 3, every F-algebra homomorphism $K \otimes_F D \to F_s$ extends in 9 different ways to A, so A is an étale F-algebra. Abusing notation, we also denote by s and t the images in A of the indeterminates. Straightforward computations show that σ and ρ extend to automorphisms of A by letting

$$\sigma(s) = st\sigma^2(a)^{-1}\,, \quad \sigma(t) = t\,, \quad \rho(s) = s^{-1}\,, \quad \rho(t) = t\,,$$

and that the extended σ, ρ satisfy $\sigma^3 = \rho^2 = \mathrm{Id}_A$ and $\rho\sigma = \sigma^2\rho$, so they generate a group G of automorphisms of A isomorphic to the symmetric group S_3. The subalgebra of A fixed under G is called the *Morley F-algebra* associated with K and a. It is denoted by $M(K, a)$.

Since G acts freely on $\mathfrak{X}(K \otimes_F D)$, it also acts freely on $\mathfrak{X}(A)$, hence

$$\dim_F M(K,a) = 9\,.$$

It readily follows from the definition that $M(K,a)$ contains the 3-dimensional étale F-algebra

$$N(K,a) = F[t]\,, \qquad \text{with } t^3 = \mathsf{N}_K(a).$$

Clearly, if $a' = \lambda k^3 a$ for some $\lambda \in F^\times$ and $k \in K^\times$, then there is an isomorphism $M(K,a') \simeq M(K,a)$ induced by $s' \mapsto s\sigma^2(k)\,\sigma(k)^{-1}$, $t' \mapsto t\lambda\,\mathsf{N}_K(k)$.

EXAMPLE 2.2. Let $K = F \times F \times F$ and $a = (a_1, a_2, a_3) \in K^\times$. Then $D \simeq F \times F$, so $K \otimes_F D \simeq F^6$. We index the primitive idempotents of $K \otimes D$ by the elements in G, so that the G-action on the primitive idempotents $(e_\tau)_{\tau \in G}$ is given by

$$\theta(e_\tau) = e_{\theta\tau} \qquad \text{for } \theta,\, \tau \in G.$$

We identify K with a subalgebra of $K \otimes D$ by

$$(x_1, x_2, x_3) = x_1(e_{\mathrm{Id}} + e_\rho) + x_2(e_\sigma + e_{\rho\sigma}) + x_3(e_{\sigma^2} + e_{\rho\sigma^2})$$

for x_1, x_2, $x_3 \in F$. Then $A \simeq F^6[s,t]$ where

$$s^3 = \frac{\sigma^2(a)}{\sigma(a)} = \frac{a_2}{a_3}e_{\mathrm{Id}} + \frac{a_3}{a_1}e_\sigma + \frac{a_1}{a_2}e_{\sigma^2} + \frac{a_3}{a_2}e_\rho + \frac{a_2}{a_1}e_{\sigma\rho} + \frac{a_1}{a_3}e_{\sigma^2\rho}$$

and

$$t^3 = a_1 a_2 a_3\,.$$

Let $r = \sum_{\tau \in G} \tau(s)\, e_\tau \in M(K,a)$. Then $r^3 = \frac{a_2}{a_3}$ and $M(K,a) = F[r,t]$. Note that $\left(\frac{r^2 t}{a_2}\right)^3 = \frac{a_1}{a_3}$, so

$$M(K,a) \simeq F\left[\sqrt[3]{\frac{a_1}{a_3}}, \sqrt[3]{\frac{a_2}{a_3}}\right] \qquad \text{and} \qquad N(K,a) \simeq F[\sqrt[3]{a_1 a_2 a_3}]\,.$$

EXAMPLE 2.3. Let K be an arbitrary cubic étale F-algebra and let $a = 1$. Let $F[\omega]$ be the quadratic étale F-algebra with $\omega^2 + \omega + 1 = 0$. By the Chinese Remainder Theorem we have

$$N(K,1) = F[t]/(t^3 - 1) \simeq F \times F[\omega]\,.$$

The corresponding orthogonal idempotents in $N(K,1)$ are

$$e_1 = \tfrac{1}{3}(1 + t + t^2) \qquad \text{and} \qquad e_2 = \tfrac{1}{3}(2 - t - t^2)\,.$$

Let $A_1 = e_1A$ and $A_2 = e_2A$, so $A = A_1 \oplus A_2$ and the G-action preserves A_1 and A_2. Let

$$e_{11} = \tfrac{1}{3}(1+s+s^2)\,e_1 \in A_1\,, \qquad e_{12} = \tfrac{1}{3}(2-s-s^2)\,e_1 \in A_1\,,$$
$$\varepsilon_1 = \tfrac{1}{3}(1+s+s^2)\,e_2 \in A_2\,, \qquad \varepsilon_2 = \tfrac{1}{3}(1+st+s^2t^2)\,e_2 \in A_2\,,$$
$$\varepsilon_3 = \tfrac{1}{3}(1+st^2+s^2t)\,e_2 \in A_2\,.$$

These elements are pairwise orthogonal idempotents, and we have

$$e_1 = e_{11} + e_{12}\,, \qquad e_2 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3\,.$$

The G-action fixes e_{11} and e_{12}, while

$$\sigma(\varepsilon_1) = \varepsilon_2\,, \qquad \sigma(\varepsilon_2) = \varepsilon_3\,, \qquad \sigma(\varepsilon_3) = \varepsilon_1\,,$$
$$\rho(\varepsilon_1) = \varepsilon_1\,, \qquad \rho(\varepsilon_2) = \varepsilon_3\,, \qquad \rho(\varepsilon_3) = \varepsilon_2\,.$$

We have $e_1t = e_1$ and $e_{11}s = e_{11}$, hence $e_{11}A \simeq K \otimes D$ and $e_{11}M(K,1) \simeq F$. On the other hand, $e_{12}s$ is a primitive cube root of unity in $e_{12}M(K,1)$. It is fixed under σ and $\rho(e_{12}s) = e_{12}s^{-1}$. Therefore we have

$$e_{12}A \simeq K \otimes D \otimes F[\omega] \quad \text{and} \quad e_{12}M(K,1) \simeq (D \otimes F[\omega])^\rho\,,$$

where ρ acts non-trivially on D and $F[\omega]$. The quadratic étale algebra $(D \otimes F[\omega])^\rho$ is the *composite* of D and $F[\omega]$ in the group of quadratic étale F-algebras, see [9, Prop. 3.11]. It is denoted by $D * F[\omega]$. Finally, we have an isomorphism $K \otimes F[\omega] \simeq e_2M(K,1)$ by mapping $x \in K$ to $x\varepsilon_1 + \sigma(x)\,\varepsilon_2 + \sigma^2(x)\,\varepsilon_3$ and ω to $e_2\,t$, so

$$M(K,1) \simeq F \times (D * F[\omega]) \times (K \otimes F[\omega])\,.$$

Under this isomorphism, the inclusion $N(K,1) \hookrightarrow M(K,1)$ is the map

$$F \times F[\omega] \to F \times (D * F[\omega]) \times (K \otimes F[\omega])\,, \qquad (x,y) \mapsto (x,x,y)\,.$$

In particular, if F contains a cube root of unity, then $F[\omega] \simeq F \times F$ and

$$M(K,1) \simeq F \times D \times K \times K\,.$$

The inclusion $N(K,1) \hookrightarrow M(K,1)$ is then given by

$$F \times F \times F \to F \times D \times K \times K\,, \qquad (x,y,z) \mapsto (x,x,y,z)\,.$$

Details are left to the reader.

In the rest of this section, we show how the Γ-set $\mathfrak{X}\big(M(K,a)\big)$ can be characterized as the fibre of a certain (ramified) covering of the projective plane.

Viewing K as an F-vector space, we may consider the projective plane $\mathbf{P}_K$, whose points over the separable closure F_s are

$$\mathbf{P}_K(F_s) = \{x \cdot F_s^\times \mid x \in K \otimes_F F_s,\ x \neq 0\}.$$

Let

$$\text{(2.1)} \quad \pi\colon \mathbf{P}_K(F_s) \to \mathbf{P}_K(F_s), \quad x \cdot F_s^\times \mapsto x^3 \cdot F_s^\times \quad \text{for } x \in K \otimes F_s,\ x \neq 0.$$

We show in Theorem 2.6 below that there is an isomorphism of Γ-sets

$$\mathfrak{X}\big(M(K,a)\big) \simeq \pi^{-1}(a \cdot F_s^\times) \quad \text{for } a \in K^\times.$$

In view of the anti-equivalence between Et_F and Set_Γ, this result characterizes the Morley algebra $M(K,a)$ up to isomorphism.

Until the end of this section, we fix $a \in K^\times$ and denote $M(K,a)$ simply by M. We identify $K \otimes M$ with the subalgebra of A fixed under ρ.

LEMMA 2.4. *There exists $u \in (K \otimes M)^\times$ such that $s = \sigma^2(u)\,\sigma(u)^{-1}$.*

Proof. Define a map $c\colon G \to A^\times$ by

$$c(\mathrm{Id}) = c(\sigma^2\rho) = 1, \qquad c(\sigma) = c(\rho) = s, \qquad c(\sigma^2) = c(\sigma\rho) = \sigma^2(s)^{-1}.$$

Computation shows that $s\sigma(s)\,\sigma^2(s) = 1$, and it follows that c is a 1-cocycle. Proposition 1.1 yields an element $v \in A^\times$ such that $c(\tau) = v\tau(v)^{-1}$ for all $\tau \in G$; in particular, we have

$$s = v\sigma(v)^{-1} = v\rho(v)^{-1}.$$

Let $u = \sigma^2(v)^{-1}$. The equations above yield

$$s = \sigma^2(u)\,\sigma(u)^{-1} \quad \text{and} \quad \rho(u) = u.$$

Therefore $u \in K \otimes M$, and this element satisfies the condition.

LEMMA 2.5. *The set $\pi^{-1}(a \cdot F_s^\times)$ has 9 elements if it is non-empty.*

Proof. Suppose $x_0 \in K \otimes F_s$ is such that $x_0^3 \cdot F_s^\times = a \cdot F_s^\times$. Then the map $y \cdot F_s^\times \mapsto x_0 y \cdot F_s^\times$ defines a bijection between $\pi^{-1}(1 \cdot F_s^\times)$ and $\pi^{-1}(a \cdot F_s^\times)$, so it suffices to show that $\left|\pi^{-1}(1 \cdot F_s^\times)\right| = 9$. Identify $K \otimes F_s = F_s \times F_s \times F_s$, and let $\omega \in F_s^\times$ be a primitive cube root of unity. To simplify notation, write

$(z_1 : z_2 : z_3) = (z_1, z_2, z_3) \cdot F_s^\times$ for $z_1, z_2, z_3 \in F_s$. It is easy to check that $\pi^{-1}(1 \cdot F_s^\times)$ consists of the following elements:

$$\begin{array}{lll} (1:1:1), & (1:\omega:\omega^2), & (1:\omega^2:\omega), \\ (1:1:\omega), & (1:\omega:1), & (\omega:1:1), \\ (1:1:\omega^2), & (1:\omega^2:1), & (\omega^2:1:1). \end{array}$$

Each $\xi \in \mathfrak{X}(M)$ extends uniquely to a K-algebra homomorphism

$$\widehat{\xi}\colon K \otimes_F M \to K \otimes_F F_s .$$

THEOREM 2.6 (Rost). *Let $u \in (K \otimes M)^\times$ be such that $\sigma^2(u)\,\sigma(u)^{-1} = s$. The map $\xi \mapsto \widehat{\xi}(u) \cdot F_s^\times$ defines an isomorphism of Γ-sets*

$$\Phi\colon \mathfrak{X}(M) \xrightarrow{\sim} \pi^{-1}(a \cdot F_s^\times) .$$

Proof. If $u \in (K \otimes M)^\times$ satisfies $\sigma^2(u)\,\sigma(u)^{-1} = s$, then

$$\sigma^2(u^3)\,\sigma(u^3)^{-1} = s^3 = \sigma^2(a)\,\sigma(a)^{-1} ,$$

so $a^{-1}u^3$ is fixed under σ, hence $a^{-1}u^3 \in M^\times$. Therefore $a^{-1}\widehat{\xi}(u)^3 \in F_s^\times$, hence $\widehat{\xi}(u) \cdot F_s^\times$ lies in $\pi^{-1}(a \cdot F_s^\times)$.

Note that the map Φ does not depend on the choice of u: indeed, u is determined uniquely up to a factor in $M^\times$, and for $m \in M^\times$ we have $\widehat{\xi}(um) = \widehat{\xi}(u)\,\xi(m)$, so $\widehat{\xi}(um) \cdot F_s^\times = \widehat{\xi}(u) \cdot F_s^\times$.

It is clear from the definition that the map Φ is Γ-equivariant. Since $|\mathfrak{X}(M)| = \left|\pi^{-1}(a \cdot F_s^\times)\right| = 9$, it suffices to show that Φ is injective to complete the proof. Extending scalars, we may assume that $K \simeq F \times F \times F$, and use the notation of Example 2.2. Then, up to a factor in $M^\times$, we have

$$\begin{aligned} u &= \sigma^2\rho(s)\,e_{\mathrm{Id}} + \sigma(s)\,e_\sigma + e_{\sigma^2} + \sigma(s)\,e_\rho + e_{\sigma\rho} + \sigma^2\rho(s)\,e_{\sigma^2\rho} \\ &= \frac{r^2t}{a_2}(e_{\mathrm{Id}} + e_\rho) + r(e_\sigma + e_{\sigma^2\rho}) + (e_{\sigma^2} + e_{\sigma\rho}) \\ &= \left(\frac{r^2t}{a_2}, r, 1\right) \in K \otimes M = M \times M \times M . \end{aligned}$$

If $\xi, \eta \in \mathfrak{X}(M)$ satisfy $\widehat{\xi}(u) \cdot F_s^\times = \widehat{\eta}(u) \cdot F_s^\times$, then $\xi\left(\frac{r^2t}{a_2}\right) = \eta\left(\frac{r^2t}{a_2}\right)$ and $\xi(r) = \eta(r)$. Since M is generated by r and t, it follows that $\xi = \eta$.

REMARK 2.7. As pointed out by Rost [10], the map π factors through

$$W(F_s) = \{(\lambda, x)\cdot F_s^\times \mid \lambda^3 = \mathsf{N}_K(x)\} \subseteq \mathbf{P}_{F\times K}(F_s) \ :$$

we have $\pi = \pi_1 \circ \pi_2$, where

$$\pi_2\colon \mathbf{P}_K(F_s) \to W(F_s)\,, \qquad x\cdot F_s^\times \mapsto (\mathsf{N}_K(x), x^3)\cdot F_s^\times$$

and

$$\pi_1\colon W(F_s) \to \mathbf{P}_K(F_s)\,, \qquad (\lambda, x)\cdot F_s^\times \mapsto x\cdot F_s^\times\,.$$

There is a commutative diagram

$$\begin{array}{ccc} \mathfrak{X}\big(M(K,a)\big) & \xrightarrow{\ \Phi\ } & \mathbf{P}_K(F_s) \\ {\scriptstyle \mathfrak{X}(i)}\big\downarrow & & \big\downarrow{\scriptstyle \pi_2} \\ \mathfrak{X}\big(N(K,a)\big) & \xrightarrow{\ \Phi'\ } & W(F_s) \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi_1} \\ \mathfrak{X}(F) & \xrightarrow{\ \Phi''\ } & \mathbf{P}_K(F_s)\,, \end{array}$$

where $\mathfrak{X}(i)$ is the map functorially associated to the inclusion

$$i\colon N(K,a) \hookrightarrow M(K,a)$$

and Φ'' maps the unique element of $\mathfrak{X}(F)$ to $a\cdot F_s^\times$. The induced map Φ' is an isomorphism of Γ-sets

$$\Phi'\colon\ \mathfrak{X}\big(N(K,a)\big) \xrightarrow{\ \sim\ } \pi_1^{-1}(a\cdot F_s^\times)\,.$$

3. INFLECTION POINT CONFIGURATIONS

Let V be a 3-dimensional vector space over F. Let $\mathsf{S}^3(V^*)$ be the third symmetric power of the dual space V^*, i.e., the space of cubic forms on V. A cubic form $f \in \mathsf{S}^3(V^*)$ is called *triangular* if its zero set in the projective plane $\mathbf{P}_V(F_s)$ defines a triangle or, equivalently, if there exist linearly independent linear forms φ_1, φ_2, $\varphi_3 \in V^* \otimes_F F_s$ such that $f = \varphi_1\varphi_2\varphi_3$ in $\mathsf{S}^3(V^* \otimes F_s)$. The *sides of the triangle* are the zero sets of φ_1, φ_2, and φ_3; they form a 3-element Γ-set $\mathfrak{S}(f)$.

PROPOSITION 3.1. *Let $f \in \mathsf{S}^3(V^*)$ be a triangular cubic form and let K be the cubic étale F-algebra such that $\mathfrak{X}(K) \simeq \mathfrak{S}(f)$. Then we may identify the F-vector spaces V and K so as to identify f with a multiple of the norm form of K,*

$$f = \lambda\,\mathsf{N}_K \quad \text{for some } \lambda \in F^\times.$$

In particular, the Γ-action on $\mathfrak{S}(f)$ is trivial if and only if f factors into a product of three independent linear forms in V^*.

Proof. Let $f = \varphi_1\varphi_2\varphi_3$ for some linearly independent linear forms $\varphi_1, \varphi_2, \varphi_3 \in V^* \otimes F_s$. Since ${}^\gamma\varphi_1{}^\gamma\varphi_2{}^\gamma\varphi_3 = \varphi_1\varphi_2\varphi_3$ for $\gamma \in \Gamma$, it follows by unique factorization in $\mathsf{S}^3(V^*)$ that there exist a permutation π_γ of $\{1,2,3\}$ and scalars $\lambda_{\pi_\gamma(i),\gamma} \in F_s^\times$ such that

$${}^\gamma\varphi_i = \lambda_{\pi_\gamma(i),\gamma}\varphi_{\pi_\gamma(i)} \quad \text{for } i = 1,\ 2,\ 3.$$

Since ${}^{\gamma\delta}\varphi_i = {}^\gamma({}^\delta\varphi_i)$ for $\gamma, \delta \in \Gamma$, we have

$$\lambda_{\pi_{\gamma\delta}(i),\gamma\delta}\,\varphi_{\pi_{\gamma\delta}(i)} = \gamma(\lambda_{\pi_\delta(i),\delta})\,\lambda_{\pi_\gamma\pi_\delta(i),\gamma}\,\varphi_{\pi_\gamma\pi_\delta(i)},$$

hence $\pi_{\gamma\delta} = \pi_\gamma\pi_\delta$ and

$$\lambda_{\pi_{\gamma\delta}(i),\gamma\delta} = \gamma(\lambda_{\pi_\delta(i),\delta})\,\lambda_{\pi_\gamma\pi_\delta(i),\gamma}. \tag{3.1}$$

The Γ-set $\mathfrak{S}(f)$ is $\{1,2,3\}$ with the Γ-action $\gamma \mapsto \pi_\gamma$; therefore we may identify K with the F-algebra of Γ-equivariant maps

$$K = \mathrm{Map}(\{1,2,3\}, F_s)^\Gamma.$$

For $\gamma \in \Gamma$, define $a_\gamma \in \mathrm{Map}(\{1,2,3\}, F_s^\times) = (K \otimes F_s)^\times$ by

$$a_\gamma(i) = \lambda_{i,\gamma}.$$

Clearly, $a_\gamma = 1$ if γ fixes φ_1, φ_2, and φ_3; moreover, by (3.1) we have $a_\gamma{}^\gamma a_\delta = a_{\gamma\delta}$ for $\gamma, \delta \in \Gamma$, hence $(a_\gamma)_{\gamma\in\Gamma}$ is a continuous 1-cocycle. By Hilbert's Theorem 90 [8, (29.2)], we have $H^1(\Gamma, (K \otimes F_s)^\times) = 1$, hence there exists $b \in \mathrm{Map}(\{1,2,3\}, F_s^\times)$ such that $a_\gamma = b\,{}^\gamma b^{-1}$ for all $\gamma \in \Gamma$. For $i = 1,\ 2,\ 3$, let $\psi_i = b(i)\,\varphi_i \in V^* \otimes F_s$. Let also

$$\lambda = \big(b(1)b(2)b(3)\big)^{-1}.$$

Computation shows that ${}^\gamma\psi_i = \psi_{\pi_\gamma(i)}$ for $\gamma \in \Gamma$ and $i = 1,\ 2,\ 3$, and $f = \lambda\psi_1\psi_2\psi_3$ in $\mathsf{S}^3(V^* \otimes F_s)$, hence $\lambda \in F^\times$. Define

$$\Theta\colon V \otimes F_s \to \mathrm{Map}(\{1,2,3\}, F_s) = K \otimes F_s$$

by

$$\Theta(x)\colon i \mapsto \psi_i(x) \quad \text{for } i = 1,2,3 \text{ and } x \in V \otimes F_s.$$

Since ψ_1, ψ_2, ψ_3 are linearly independent, Θ is an F_s-vector space isomorphism. It restricts to an isomorphism of F-vector spaces $V \xrightarrow{\sim} K$ under which f is identified with $\lambda\,\mathsf{N}_K$.

Now, let $\mathfrak{I} \subseteq \mathbf{P}_V(F_s)$ be a 9-point set that has the characteristic property of the set of inflection points of a nonsingular cubic curve: the line through any two distinct points of $\mathfrak{I}$ passes through exactly one third point of $\mathfrak{I}$. Let $\mathfrak{L}$ be the set of lines in $\mathbf{P}_V(F_s)$ that are incident to three points of $\mathfrak{I}$. This set has 12 elements, and $\mathfrak{I}$, $\mathfrak{L}$ form an incidence geometry that is isomorphic to the affine plane over the field with three elements, see [7, § 11.1]. In particular, there is a partition of $\mathfrak{L}$ into four subsets $\mathfrak{T}_0, \dots, \mathfrak{T}_3$ of three lines, which we call *triangles*, with the property that each point of $\mathfrak{I}$ is incident to one and only one line of each triangle.

Assume $\mathfrak{I}$ is stable under the action of Γ, and Γ preserves the triangle $\mathfrak{T}_0$. Let K be the cubic étale F-algebra whose Γ-set $\mathfrak{X}(K)$ is isomorphic to $\mathfrak{T}_0$. By Proposition 3.1, we may identify V with K in such a way that the union of the lines in $\mathfrak{T}_0$ is the zero set of the norm N_K.

THEOREM 3.2. *There exists $a \in K^\times$ such that the Γ-set of vertices of the triangles $\mathfrak{T}_1$, $\mathfrak{T}_2$, $\mathfrak{T}_3$ is $\pi^{-1}(a \cdot F_s^\times)$, where $\pi\colon \mathbf{P}_K(F_s) \to \mathbf{P}_K(F_s)$ is defined in* (2.1). *The set $\mathfrak{I}$ is the set of inflection points of the cubics in the pencil spanned by the forms $\mathsf{T}_K(a^{-1}X^3)$ and $\mathsf{N}_K(X)$, and we have isomorphisms of Γ-sets*

$$\mathfrak{L} \simeq \mathfrak{X}(K) \coprod \mathfrak{X}\big(M(K,a)\big), \qquad \{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3\} \simeq \mathfrak{X}\big(N(K,a)\big).$$

Proof. Fix an isomorphism $K \otimes F_s \simeq F_s \times F_s \times F_s$, and write simply $(x_1 : x_2 : x_3)$ for $(x_1, x_2, x_3) \cdot F_s^\times$. The sides of $\mathfrak{T}_0$ are then the lines with equation $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$. Let $\mathfrak{I} = \{p_1, \dots, p_9\}$. We label the points so that the incidence relations can be read from the representation of the affine plane over $\mathbf{F}_3$ in Figure 1.

Say the line through p_1, p_2, p_3 is $x_1 = 0$, and the line through p_4, p_5, p_6 is $x_2 = 0$. We can then find u_1, u_2, u_3, $v \in F_s^\times$ such that

$$p_i = (0 : u_i : 1) \quad \text{for } i = 1,2,3, \quad \text{and} \quad p_4 = (1 : 0 : v).$$

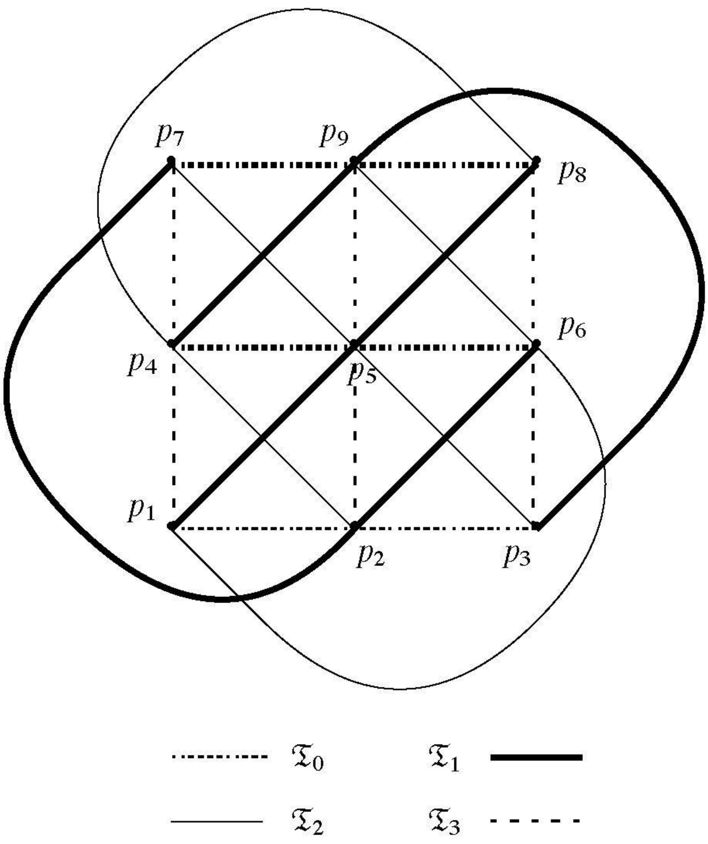

FIGURE 1
Incidence relations on $\mathfrak{I}$

Since p_7 lies at the intersection of $x_3 = 0$ with the line through p_1 and p_4, we have

$$p_7 = (1 : -u_1 v : 0).$$

Similarly,

$$p_8 = (1 : -u_2 v : 0) \quad \text{and} \quad p_9 = (1 : -u_3 v : 0).$$

Finally, since p_5 (resp. p_6) lies at the intersection of $x_2 = 0$ with the line through p_1 and p_8 (resp. p_9), we have

$$p_5 = (u_1 : 0 : u_2 v) \quad \text{and} \quad p_6 = (u_1 : 0 : u_3 v).$$

Collinearity of the points p_2, p_6, p_7 (resp. p_2, p_5, p_9 ; resp. p_3, p_6, p_8) yields

$$u_1^2 = u_2 u_3 , \qquad (\text{resp. } u_2^2 = u_1 u_3 \text{ ; resp. } u_3^2 = u_1 u_2).$$

Therefore

$$u_1^3 = u_2^3 = u_3^3 = u_1u_2u_3\,.$$

Since u_1, u_2, u_3 are pairwise distinct, it follows that there is a primitive cube root of unity $\omega \in F_s$ such that

$$u_2 = \omega u_1 \qquad \text{and} \qquad u_3 = \omega^2 u_1\,.$$

Straightforward computations yield the vertices of the triangles $\mathfrak{T}_1$, $\mathfrak{T}_2$, $\mathfrak{T}_3$:

$$\begin{array}{llll}
\mathfrak{T}_1: & q_1 = (1 : \omega^2 u_1 v : -v)\,, & q_1' = (1 : u_1 v : -\omega^2 v)\,, & q_1'' = (\omega^2 : u_1 v : -v)\,,\\
\mathfrak{T}_2: & q_2 = (\omega : u_1 v : -v)\,, & q_2' = (1 : u_1 v : -\omega v)\,, & q_2'' = (1 : \omega u_1 v : -v)\,,\\
\mathfrak{T}_3: & q_3 = (1 : \omega u_1 v : -\omega^2 v)\,, & q_3' = (\omega^2 : \omega u_1 v : -v)\,, & q_3'' = (1 : u_1 v : -v)\,.
\end{array}$$

Let $a_0 = (1, u_1^3 v^3, -v^3) \in (K \otimes F_s)^\times$. It is readily verified that

$$\{q_1, q_1', q_1'', q_2, q_2', q_2'', q_3, q_3', q_3''\} = \pi^{-1}(a_0 \cdot F_s^\times)\,.$$

Since $\mathfrak{I}$ is stable under the action of Γ, the point $a_0 \cdot F_s^\times$ is fixed under Γ, hence for $\gamma \in \Gamma$ there exists $\lambda_\gamma \in F_s^\times$ such that

$$\gamma(a_0) = a_0 \lambda_\gamma \qquad \text{in } K \otimes F_s\,.$$

Then $(\lambda_\gamma)_{\gamma\in\Gamma}$ is a continuous 1-cocycle of Γ in $F_s^\times$. Hilbert's Theorem 90 yields an element $\mu \in F_s^\times$ such that $\lambda_\gamma = \mu\gamma(\mu)^{-1}$ for all $\gamma \in \Gamma$. Then for $a = a_0\mu$ we have $a_0 \cdot F_s^\times = a \cdot F_s^\times$ and $\gamma(a) = a$ for all $\gamma \in \Gamma$, hence $a \in K^\times$.

The inflection points of the cubics in the pencil spanned by $\mathsf{T}_K(a^{-1}X^3)$ and $\mathsf{N}_K(X)$ are the points $(x_1 : x_2 : x_3)$ such that

$$\begin{cases} x_1^3 + (u_1 v)^{-3} x_2^3 - v^{-3} x_3^3 = 0\,,\\ x_1 x_2 x_3 = 0\,. \end{cases}$$

The solutions of this system are exactly the points $p_1, \ldots, p_9$.

Finally, the Γ-set of sides of the triangle $\mathfrak{T}_0$ is isomorphic to $\mathfrak{X}(K)$ by hypothesis, and the map that associates to each side of a triangle its opposite vertex defines an isomorphism between the set of sides of $\mathfrak{T}_1$, $\mathfrak{T}_2$, $\mathfrak{T}_3$ and the set $\{q_1, \ldots, q_3''\} = \pi^{-1}(a \cdot F_s^\times)$. By Theorem 2.6, we have $\pi^{-1}(a \cdot F_s^\times) \simeq \mathfrak{X}\big(M(K,a)\big)$, hence

$$\mathfrak{L} \simeq \mathfrak{X}(K) \coprod \mathfrak{X}\big(M(K,a)\big)\,.$$

This isomorphism induces an isomorphism

$$\{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3\} \simeq \mathfrak{X}\big(N(K,a)\big)\,,$$

which can be made explicit by the following observation: the triangular cubic forms in the pencil spanned by $\mathsf{T}_K(a^{-1}X^3)$ and $N_K(X)$ are the scalar multiples of $\mathsf{N}_K(X)$ (whose zero set is the triangle $\mathfrak{T}_0$) and of $\mathsf{T}_K(a^{-1}X^3) - 3z\,\mathsf{N}_K(X)$, where $z \in F_s^\times$ is such that $z^3 = \mathsf{N}_K(a^{-1})$. The zero set of the latter form is $\mathfrak{T}_1$, $\mathfrak{T}_2$ or $\mathfrak{T}_3$ depending on the choice of z, and the three values of z are in one-to-one correspondence with the elements in the fibre of the map π_1 in Remark 2.7.

4. Normal forms of ternary cubics

Let V be a 3-dimensional vector space over F and let $f \in \mathsf{S}^3(V^*)$ be a nonsingular cubic form. Recall from the introduction the notation $\mathfrak{I}(f)$ (resp. $\mathfrak{L}(f)$, resp. $\mathfrak{T}(f)$) for the set of inflection points (resp. inflectional lines, resp. inflectional triangles) of f. The following result is a direct application of Theorem 3.2:

Corollary 4.1. *Let K be a cubic étale F-algebra. The following conditions are equivalent:*

(i) *f is isometric to a cubic form $\mathsf{T}_K(a^{-1}X^3) - 3\lambda\,\mathsf{N}_K(X)$ for some unit $a \in K^\times$ and some scalar $\lambda \in F$;*

(ii) *Γ has a fixed point $\mathfrak{T}_0 \in \mathfrak{T}(f)$ with $\mathfrak{T}_0 \simeq \mathfrak{X}(K)$ (as Γ-sets of 3 elements).*

When these conditions hold, we have

$$\mathfrak{L}(f) \simeq \mathfrak{X}(K) \coprod \mathfrak{X}\big(M(K,a)\big) \quad \text{and} \quad \mathfrak{T}(f) \simeq \{\mathfrak{T}_0\} \coprod \mathfrak{X}\big(N(K,a)\big).$$

Proof. If $f(X) = \mathsf{T}_K(a^{-1}X^3) - 3\lambda\,\mathsf{N}_K(X)$, then computation shows that the zero set of N_K is an inflectional triangle of f. This triangle is clearly preserved under the Γ-action. Conversely, if $\mathfrak{T}_0 \in \mathfrak{T}(f)$ is preserved under the Γ-action and K is the cubic étale F-algebra such that $\mathfrak{X}(K) \simeq \mathfrak{T}_0$, Theorem 3.2 yields an element $a \in K^\times$ such that the forms $\mathsf{T}_K(a^{-1}X^3)$ and $\mathsf{N}_K(X)$ span the pencil of cubics whose set of inflection points is $\mathfrak{I}(f)$.

Applying Corollary 4.1 in the case where F is a finite field yields a direct proof of the following result from [7, p. 276]:

Corollary 4.2. *Suppose F is a finite field with q elements. For any nonsingular cubic form f, the number of inflectional triangles of f defined over F is 0, 1, or 4 if $q \equiv 1 \bmod 3$; it is 0 or 2 if $q \equiv -1 \bmod 3$.*

Proof. Since F is finite, the action of Γ on $\mathfrak{T}(f)$ factors through a cyclic group. If there is at least one fixed triangle $\mathfrak{T}_0$, then Corollary 4.1 yields a decomposition

$$\mathfrak{T}(f) \simeq \{\mathfrak{T}_0\} \coprod \mathfrak{X}\big(N(K,a)\big)\,,$$

where $N(K,a) = F[t]$ with $t^3 = \mathsf{N}_K(a)$. If $N(K,a)$ is a field, then it must be a cyclic extension of F, hence F contains a primitive cube root of unity and therefore $q \equiv 1 \bmod 3$. Similarly, if $N(K,a) \simeq F\times F\times F$, then F contains a primitive cube root of unity. Thus, if $q \equiv -1 \bmod 3$, the Γ-action on $\mathfrak{T}(f)$ has either 0 or 2 fixed points. If $q \equiv 1 \bmod 3$ then F contains a primitive cube root of unity and either the polynomial $x^3 - \mathsf{N}_K(a)$ is irreducible or it splits into linear factors. Therefore the Γ-action on $\mathfrak{T}(f)$ has either 0, 1 or 4 fixed points.

We next spell out the special case of Corollary 4.1 where the cubic étale F-algebra K is the split algebra $F\times F\times F$:

COROLLARY 4.3. *There is a basis of V in which f takes the generalized Hesse normal form $a_1x_1^3 + a_2x_2^3 + a_3x_3^3 - 3\lambda x_1x_2x_3$ for some a_1, a_2, $a_3 \in F^\times$ and $\lambda \in F$ if and only if Γ has a fixed point $\mathfrak{T}_0 \in \mathfrak{T}(f)$ and acts trivially on $\mathfrak{T}_0$ (viewed as a 3-element subset of $\mathfrak{L}(f)$).*

EXAMPLE 4.4. Let K be a cubic étale F-algebra and let $f(X) = \mathsf{T}_K(X^3)$. By Corollary 4.1 we have

$$\mathfrak{L}(f) \simeq \mathfrak{X}(K) \coprod \mathfrak{X}\big(M(K,1)\big) \quad\text{and}\quad \mathfrak{T}(f) \simeq \{\mathfrak{T}_0\} \coprod \mathfrak{X}\big(N(K,1)\big)\,.$$

The Γ-sets $\mathfrak{X}\big(M(K,1)\big)$ and $\mathfrak{X}\big(N(K,1)\big)$ are determined in Example 2.3:

$$\mathfrak{X}\big(M(K,1)\big) \simeq \mathfrak{X}(F) \coprod \mathfrak{X}(D * F[\omega]) \coprod \mathfrak{X}(K\otimes F[\omega])$$

and

$$\mathfrak{X}\big(N(K,1)\big) \simeq \mathfrak{X}(F) \coprod \mathfrak{X}(F[\omega])\,.$$

The map $\mathfrak{X}(i)\colon \mathfrak{X}\big(M(K,1)\big) \to \mathfrak{X}\big(N(K,1)\big)$ functorially associated to the inclusion $i\colon N(K,1) \hookrightarrow M(K,1)$ maps $\mathfrak{X}(F)\coprod\mathfrak{X}(D * F[\omega])$ to $\mathfrak{X}(F)$ and $\mathfrak{X}(K\otimes F[\omega])$ to $\mathfrak{X}(F[\omega])$.

If $K \simeq F\times F\times F$, then $f(x_1,x_2,x_3) = x_1^3 + x_2^3 + x_3^3$ so f has a Hesse normal form. If $K \not\simeq F\times F\times F$, then the Γ-action on $\mathfrak{X}(K)$, hence also on $\mathfrak{X}(K\otimes F[\omega])$, is nontrivial. Therefore it follows from Corollary 4.3 that f has a generalized Hesse normal form over F if and only if the Γ-action on $\mathfrak{X}(D * F[\omega])$ is trivial. This happens if and only if $D \simeq F[\omega]$, which is equivalent to $K \simeq F[\sqrt[3]{d}]$ for some $d \in F^\times$, by [8, (18.32)]. Indeed, for $X = x_1 + x_2\sqrt[3]{d} + x_3\sqrt[3]{d^2}$, computation yields

$$f(X) = 3(x_1^3 + dx_2^3 + d^2x_3^3 + 6dx_1x_2x_3).$$

Corollary 4.3 applies in particular when F is the field $\mathbf{R}$ of real numbers:

COROLLARY 4.5. *Every nonsingular cubic form over* $\mathbf{R}$ *can be reduced to a generalized Hesse normal form.*

Proof. It is clear from the Weierstrass normal form that every nonsingular cubic over $\mathbf{R}$ has three real collinear inflection points, see [3, Prop. 14, p. 305]. The inflectional line through these points is fixed under Γ, hence the Γ-action on $\mathfrak{T}(f)$ has at least one fixed point. The same argument as in Corollary 4.2 then shows that Γ has exactly two fixed points in $\mathfrak{T}(f)$. Let $\mathfrak{T}_0$, $\mathfrak{T}_1 \in \mathfrak{T}(f)$ be the fixed inflectional triangles. Assume the Γ-action on $\mathfrak{T}_0$ (viewed as a 3-element set) is not trivial, hence $K \simeq \mathbf{R}\times\mathbf{C}$ in the notation of Corollary 4.1; we shall prove that the Γ-action on $\mathfrak{T}_1$ is trivial. By Corollary 4.1, there is a unit $a = (a_1, a_2) \in \mathbf{R}\times\mathbf{C}$ such that

$$\mathfrak{L}(f) \simeq \mathfrak{X}(\mathbf{R}\times\mathbf{C}) \coprod \mathfrak{X}\big(M(\mathbf{R}\times\mathbf{C}, a)\big).$$

By Theorem 2.6, we have an isomorphism of Γ-sets

$$\Phi\colon\ \mathfrak{X}\big(M(\mathbf{R}\times\mathbf{C}, a)\big) \xrightarrow{\sim} \pi^{-1}(a\cdot\mathbf{C}^\times) \subset \mathbf{P}_{\mathbf{R}\times\mathbf{C}}(\mathbf{C}).$$

We identify $(\mathbf{R}\times\mathbf{C})\otimes_{\mathbf{R}}\mathbf{C}$ with $\mathbf{C}\times\mathbf{C}\times\mathbf{C}$ by mapping $(r,x)\otimes y$ to $(ry, xy, \overline{x}y)$ for $r\in\mathbf{R}$ and x, $y\in\mathbf{C}$. Then the Γ-action on $\mathbf{P}_{\mathbf{R}\times\mathbf{C}} = \mathbf{P}^3_{\mathbf{C}}$ is such that the complex conjugation $^{-}$ acts by

$$(x_1 : x_2 : x_3) \mapsto (\overline{x_1} : \overline{x_3} : \overline{x_2}).$$

If $\xi\in\mathbf{R}$ and $\eta\in\mathbf{C}$ satisfy $\xi^3 = a_1$ and $\eta^3 = a_2$, and if $\omega\in\mathbf{C}$ is a primitive cube root of unity, then the proof of Lemma 2.5 shows that $\pi^{-1}(a\cdot\mathbf{C}^\times)$ consists of the following elements:

$$\begin{array}{lll}
(\xi:\eta:\overline{\eta}), & (\xi:\omega\eta:\overline{\omega\eta}), & (\xi:\overline{\omega}\eta:\omega\overline{\eta}),\\
(\xi:\eta:\omega\overline{\eta}), & (\xi:\omega\eta:\overline{\eta}), & (\omega\xi:\eta:\overline{\eta}),\\
(\xi:\eta:\overline{\omega\eta}), & (\xi:\overline{\omega}\eta:\overline{\eta}), & (\overline{\omega}\xi:\eta:\overline{\eta}).
\end{array}$$

The three points in the first row of this table are fixed under the Γ-action, whereas the Γ-action interchanges the second and third row. Therefore the first row corresponds to $\mathfrak{T}_1$ under Φ, and the proof is complete.

When the conditions in Corollary 4.1 do not hold, we may still consider the 4-dimensional étale F-algebra $T(f)$ such that $\mathfrak{X}\big(T(f)\big) = \mathfrak{T}(f)$, and the 12-dimensional étale F-algebra $L(f)$ such that $\mathfrak{X}\big(L(f)\big) = \mathfrak{L}(f)$, which is a cubic étale extension of $T(f)$. The separability idempotent $e \in T(f) \otimes_F T(f)$ satisfies $e \cdot \big(T(f) \otimes T(f)\big) \simeq T(f)$, and hence yields a decomposition

$$T(f) \otimes_F T(f) \simeq T(f) \times T(f)_0$$

for some cubic algebra $T(f)_0$ over $T(f)$. Likewise, multiplication in $L(f)$ yields an isomorphism

$$e \cdot \big(L(f) \otimes T(f)\big) \simeq L(f)\,;$$

hence

$$L(f) \otimes_F T(f) \simeq L(f) \times L(f)_0$$

for some cubic algebra $L(f)_0$ over $T(f)_0$. By functoriality of the construction of L and T, the cubic form $f_{T(f)}$ over $V \otimes_F T(f)$ obtained from f by scalar extension to $T(f)$ satisfies

$$L(f_{T(f)}) \simeq L(f) \otimes_F T(f) \quad \text{and} \quad T(f_{T(f)}) \simeq T(f) \otimes_F T(f)\,.$$

Corollary 4.1 applied to $f_{T(f)}$ shows that $f_{T(f)}$ is isometric to

$$\mathsf{T}_{L(f)}(a^{-1}X^3) - 3\lambda\, \mathsf{N}_{L(f)}(X)$$

for some $\lambda \in T(f)^\times$ and some $a \in L(f)^\times$ such that $L(f)_0$ is a Morley $T(f)$-algebra $L(f)_0 \simeq M(L(f), a)$.

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(Reçu le 8 juin 2008)

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