3.10 Proof of Theorem 1.8

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The following important theorem shows that this action extends to Q_m.

THEOREM 3.22 ([BEG]). *There exists a unique representation of the algebra eH_me on Q_m in which an element $q \in \mathbf{C}[\mathfrak{h}]^W$ acts by multiplication and an element $q \in \mathbf{C}[\mathfrak{h}^*]^W$ by L_q.*

Proof. Since by Proposition 3.5, L_q preserves Q_m, we get a uniquely defined representation of the subalgebra of eH_me generated by $\mathbf{C}[\mathfrak{h}]^W$ and $\mathbf{C}[\mathfrak{h}^*]^W$ on Q_m. The result now follows from Theorem 3.21. $\square$

3.10 PROOF OF THEOREM 1.8

Finally we can prove Theorem 1.8.

To do this, observe that as an eH_me-module, Q_m is in the category $\mathcal{O}(eH_me)$, and $\mathbf{C}[\mathfrak{h}^*]^W$ acts locally nilpotently in Q_m (by degree arguments). We can now apply Theorem 3.18 and Theorem 3.17 and deduce that Q_m is a direct sum of modules of the form $eM(0,\tau)$. As a $\mathbf{C}[\mathfrak{h}] \rtimes \mathbf{C}[W]$-module, $M(0,\tau) = \mathbf{C}[\mathfrak{h}] \otimes \tau$. On the other hand, by Chevalley's theorem, there is an isomorphism $\mathbf{C}[\mathfrak{h}] \simeq \mathbf{C}[\mathfrak{h}]^W \otimes \mathbf{C}[W]$, commuting with the action of W and $\mathbf{C}[\mathfrak{h}]^W$. Thus we get an isomorphisms of $\mathbf{C}[\mathfrak{h}]^W$-modules

$$eM(0,\tau) \simeq (M(0,\tau))^W \simeq \mathbf{C}[\mathfrak{h}]^W \otimes (\mathbf{C}[W] \otimes \tau)^W \simeq \mathbf{C}[\mathfrak{h}]^W \otimes \tau\,,$$

proving that $eM(0,\tau)$ and hence Q_m is a free $\mathbf{C}[\mathfrak{h}]^W$-module. $\square$

EXAMPLE 3.23. For $W = \mathbf{Z}/2$ and $\mathfrak{h} = \mathbf{C}$, take the polynomials 1, x^{2m+1}. Notice that $L(1) = L(x^{2m+1}) = 0$ while $s(1) = 1$, $s(x^{2m+1}) = -x^{2m+1}$, $s \in \mathbf{Z}/2$ being the element of order two. It follows that Q_m as a eH_me-module is the direct sum of $\mathbf{C}[x^2] \oplus x^{2m+1}\mathbf{C}[x^2]$. These modules are irreducible. Moreover, $\mathbf{C}[x^2] \simeq eM(0,\mathbf{1})$, $x^{2m+1}\mathbf{C}[x^2] \simeq eM(0,\varepsilon)$, ε being the sign representation.

3.11 PROOF OF THEOREM 1.15

Let I be a nonzero two-sided ideal in $\mathcal{D}(X_m)$. First we claim that I nontrivially intersects Q_m. Indeed, otherwise let $K \in I$ be a lowest order nonzero element in I. Since the order of K is positive, there exists $f \in Q_m$ such that $[K,f] \neq 0$. Then $[K,f] \in I$ is of smaller order than K, a contradiction.

Now let $f \in Q_m$ be an element of I. Then $g = \prod_{w \in W} {}^w\!f \in I$. But g is W-invariant. This shows that the intersection J of I with the subalgebra H_m in $\mathcal{D}(X_m)$ is nonzero. But H_m is simple by Theorem 3.19, so $J = H_m$. Hence, $1 \in J \subset I$, and $I = \mathcal{D}(X_m)$. $\square$