3.2 Berest's formula for L_q

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differential operator S_m of the form $\delta_m(x)\delta_m(\partial_x)+l.o.t.$, with $\delta_m(x) = \prod_{s\in\Sigma} \alpha_s^{m_s}$ such that

$$L_q S_m = S_m q(\partial)$$

for every $q \in \mathbf{C}[\mathfrak{h}] = \mathbf{C}[q_1, \dots, q_n]$. From this, if we set $\psi(k,x) = S_m e^{(k,x)}$, we get

$$L_q \psi = S_m q(\partial)\, e^{(k,x)} = q(k)\psi\,, \tag{7}$$

$q \in \mathbf{C}[q_1, \dots, q_n]$.

We claim that equation (7) must in fact hold for all $q \in Q_m$. Indeed, near a generic point x, the functions $\psi(wk,x)$ are obviously linearly independent and satisfy (7) for symmetric q. Thus, they are a basis in the space of solutions (we know that this space is $|W|$-dimensional). Consider the matrix of L_q in this basis for any $q \in Q_m$. Since $\psi(k,x)$ is a polynomial multiplied by $e^{(k,x)}$, this matrix must be diagonal with eigenvalues $q(k)$, as desired.

EXAMPLE 3.1. As we have seen in the previous section, for $W = \mathbf{Z}/2$ and $\mathfrak{h} = \mathbf{C}$,

$$S_m = (x\partial - 2m + 1)(x\partial - 2m - 1)\cdots(x\partial - 1)\,.$$

3.2 BEREST'S FORMULA FOR L_q

We are now going to give an explicit construction of the operators L_q for any $q \in Q_m$.

Let us identify, using our W-invariant scalar product, $\mathfrak{h}$ with $\mathfrak{h}^*$, and let us choose a orthonormal basis $x_1, \dots, x_n$ in $\mathfrak{h}^*$. If $x \in \mathfrak{h}^*$, we will write D_x for the Dunkl operator relative to the vector in $\mathfrak{h}$ corresponding to x under our identification. Thus

$$L = \sum_{i=1}^{n} D_{x_i}^2\,.$$

PROPOSITION 3.2 (Berest [Be]). *If $q \in Q_m$ is a homogeneous element of degree d, then*

$$(\operatorname{ad} L)^{d+1} q = 0\,.$$

Proof. It is enough to prove that

$$((\operatorname{ad} L)^{d+1} q)\psi(k,x) = 0\,.$$

Indeed, it follows from the definition of $\psi(k,x)$ that in the ring $\mathcal{D}(U)$ this implies: $((\operatorname{ad} L)^{d+1} q)\, S_m = 0$, so that $(\operatorname{ad} L)^{d+1}\, q = 0$, since $\mathcal{D}(U)$ is a domain.

Given $q \in Q_m$, we will denote by $L_q^{(k)}$ the operator $q(D_{k_1}, \ldots, D_{k_n})$. Notice that since $\psi(k,x) = \psi(x,k)$, we have $L_q^{(k)}\psi = q(x)\psi$. Thus we deduce, for $p, q, r \in Q_m$,

$$\begin{aligned} L_q r(x) L_p \psi &= L_q r(x) p(k) \psi = p(k) L_q r(x)\, \psi \\ &= p(k) L_q L_r^{(k)} \psi = p(k) L_r^{(k)} L_q \psi = p(k) L_r^{(k)} q(k)\, \psi\,. \end{aligned}$$

It follows that

$$(\operatorname{ad} L)^{d+1} q \psi = (-1)^{d+1} (\operatorname{ad}(\sum_{i=1}^{n} k_i^2))^{d+1} L_q^{(k)} \psi\,.$$

Since L_q is a differential operator of degree d, we get $\operatorname{ad}(\sum_{i=1}^n k_i^2)^{d+1} L_q^{(k)} = 0$, as desired. □

Notice now that the operator $(adL)^d q(x)$ commutes with L. Its symbol is given by $(\operatorname{ad}\Delta)^d q(x) = 2^d d! q(\partial)$. So we deduce the following

COROLLARY 3.3 (Berest's formula, [Be]). *If $q \in Q_m$ is homogeneous of degree d, then*

$$L_q = \frac{1}{2^d d!} (\operatorname{ad} L)^d q(x)\,.$$

Proof. This is clear from Proposition 2.8, once we remark that $(\operatorname{ad} L)^d q(x)$ has the required homogeneity. □

We want to give a representation theoretical interpretation of what we have just seen. Consider the three operators

$$F = \frac{\sum_{i=1}^n x_i^2}{2}\,, \quad E = -\frac{L}{2}\,, \quad H = [E, F]\,. \tag{8}$$

It is easy to check that $[H, E] = 2E$, $[H, F] = -2F$. We deduce that the elements E, F, H span an $\mathfrak{sl}(2)$ Lie subalgebra of $\mathcal{D}(U)$. Thus $\mathfrak{sl}(2)$ acts by conjugation on $\mathcal{D}(U)$. We can then reformulate Proposition 3.2 as follows:

PROPOSITION 3.4. *Any polynomial $q \in Q_m$ of degree d is a lowest weight vector for the $\mathfrak{sl}(2)$-action of weight $-d$ and generates a finite dimensional module (necessarily of dimension $d+1$) for which L_q is a highest weight vector.*

Proof. An easy direct computation shows that

$$H = [E, F] = -\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} + C,$$

where C is a constant. Thus if q is homogeneous of degree d, we have $[H, L_q] = dL_q$.

This and the fact that $[L, L_q] = 0$, implies that L_q is a highest weight vector of weight d. Also since F is a polynomial, we deduce that $\operatorname{ad} F^{d+1} L_q = 0$, so that L_q generates a $(d+1)$-dimensional irreducible $\mathfrak{sl}(2)$-module. □

One last property about these operators is given by

PROPOSITION 3.5 ([FV]). *For any $q \in Q_m$, the operator L_q preserves Q_m.*

Proof. Let us begin by proving that L preserves Q_m.

Take $f \in Q_m$, so that for any $s \in \Sigma$, $f - {}^s f = \alpha_s^{2m_s+1} t$, $t \in \mathbf{C}[\mathfrak{h}]$. Let us start by showing that Lf is a polynomial. Clearly $Lf = \delta_*^{-1} q$, with $q \in \mathbf{C}[\mathfrak{h}]$, and $\delta_* = \prod_{s: m_s \neq 0} \alpha_s$. Since L is W-invariant, $Lf - {}^s(Lf) = L(f - {}^s f)$ is clearly divisible by $\alpha_s^{2m_s-1}$ if $m_s > 0$. In particular, it is always regular along the reflection hyperplane of s. On the other hand, since $Lf - {}^s(Lf) = \delta_*^{-1}(q + {}^s q)$, we deduce that $q + {}^s q$ is divisible by α_s if $m_s > 0$. But then $q = ((q + {}^s q) + (q - {}^s q))/2$ is divisible by α_s if $m_s > 0$, hence it is divisible by δ_*, so that Lf lies in $\mathbf{C}[\mathfrak{h}]$.

We have already remarked that $L(f - {}^s f)$ is divisible by $\alpha_s^{2m_s-1}$ if $m_s > 0$. In fact

$$L(f - {}^s f) = (L\alpha_s^{2m_s+1})t + \alpha_s^{2m_s}\tilde{t},$$

where $\tilde{t}$ is a suitable polynomial.

But since

$$L\alpha_s^{2m_s+1} = 2m_s(2m_s+1)(\alpha_s, \alpha_s)\alpha_s^{2m_s-1} - 2m_{s'}(2m_s+1)\sum_{s' \in \Sigma}(\alpha_{s'}, \alpha_s)\frac{\alpha_s^{2m_s}}{\alpha_{s'}}$$

$$= -2m_{s'}(2m_s+1)\sum_{s' \in \Sigma, s' \neq s}(\alpha_{s'}, \alpha_s)\frac{{\alpha_s}^{2m}}{\alpha_{s'}},$$

we deduce that $L(f - {}^s f)$ is divisible by $\alpha_s^{2m_s}$. On the other hand, since $L(f - {}^s f) = Lf - {}^s(Lf)$, this polynomial is either zero or it must vanish to odd order on the reflection hyperplane of s. We deduce that it must be divisible by $\alpha_s^{2m_s+1}$, proving that $Lf \in Q_m$.

We now pass to a general L_q, $q \in Q_m$. We may assume that q is homogeneous of, say, degree d. By Corollary 3.3 we have that L_q is a non zero multiple of $(adL)^d(q)$. Since both q and L preserve Q_m, our claim follows. $\square$

3.3 Differential operators on X_m

Now let us return to the algebra of differential operators $\mathcal{D}(X_m)$. Notice that $\mathcal{D}(X_m)$ contains two commutative subalgebras (both isomorphic to Q_m). The first is Q_m itself, the second is the subalgebra $Q_m^\dagger$ consisting of the differential operators of the form L_q with $q \in Q_m$. It is possible to prove

THEOREM 3.6 ([BEG]). *$\mathcal{D}(X_m)$ is generated by Q_m and $Q_m^\dagger$.*

Notice that by Corollary 3.3 we in fact have that $\mathcal{D}(X_m)$ is generated by Q_m and by L.

EXAMPLE 3.7. If $W = \mathbf{Z}/2$, $\mathfrak{h} = \mathbf{C}$ we get that $\mathcal{D}(X_m)$ is generated by the operators

$$x^2, \quad x^{2m+1}, \quad \frac{d^2}{dx^2} - \frac{2m}{x}\frac{d}{dx}.$$

Theorem 3.6 together with Proposition 3.4, imply

COROLLARY 3.8 ([BEG]). *$\mathcal{D}(X_m)$ is locally finite dimensional under the action of the Lie algebra $\mathfrak{sl}(2)$ defined in (8).*

This Corollary implies that our $\mathfrak{sl}(2)$ action on $\mathcal{D}(X_m)$ can be integrated to an action of the group $SL(2)$. In particular we have

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} q = L_q$$

for all $q \in Q_m$. This transformation is a generalization of the Fourier transform, since it reduces to the usual Fourier transform on differential operators on $\mathfrak{h}$ when $m = 0$.

EXAMPLE 3.9. If $W = \mathbf{Z}/2$, $\mathfrak{h} = \mathbf{C}$, we get that the monomials $\{x^{2i}\} \cup \{x^{2i+2m+1}\}$ are (up to constants) all lowest weight vectors for the $\mathfrak{sl}(2)$ action on $\mathcal{D}(X_m)$. x^n has weight $-n$. We deduce that $\mathcal{D}(X_m)$ is isomorphic as a $\mathfrak{sl}(2)$-module to the direct sum of the irreducible representations of dimension $n+1$ for n even or $n = 2(m+i)+1$, each with multiplicity one.