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The solution can easily be computed by differentiating the first equation and then subtracting the second, thus obtaining the new system

$$\begin{cases} \psi'' - \dfrac{2}{x}\psi' = k^2\psi\,, \\ \psi'' - (\dfrac{1}{x} + k^2x)\psi' = -k^3x\psi\,. \end{cases}$$

Taking the difference, we get the first order equation

$$\psi' = \frac{k^2x}{kx-1}\psi\,,$$

whose solution (up to constants) is given by $\psi = (kx-1)\,e^{kx}$.

In fact, one can easily calculate ψ_m for a general m.

PROPOSITION 2.12. $\psi_m(k,x) = (x\partial - 2m+1)(x\partial - 2m-1)\cdots(x\partial - 1)\,e^{kx}$.

Proof. We could use the direct method of Example 2.11, but it is more convenient to proceed differently. Namely, we have

$$(\partial^2 - \frac{2m}{x}\partial)(x\partial - 2m+1) = (x\partial - 2m+1)(\partial^2 - \frac{2(m-1)}{x}\partial)$$

as it is easy to verify directly. So using induction on m starting with $m=0$, we get

$$(\partial^2 - \frac{2m}{x}\partial)\psi_m(k,x) = (x\partial - 2m+1)(\partial^2 - \frac{2(m-1)}{x}\partial)\psi_{m-1}(k,x) = k^2\psi_m(k,x)\,,$$

and $\psi_m(k,x)$ is our solution. $\square$

3. LECTURE 3

3.1 SHIFT OPERATOR AND CONSTRUCTION OF THE BAKER-AKHIEZER FUNCTION

In Lecture 2, we have introduced the Baker-Akhiezer function $\psi(k,x)$ for the operator

$$L = \Delta - \sum_{s\in\Sigma} \frac{2c_s}{\alpha_s(x)}\partial_{\alpha_s}\,.$$

The way to construct $\psi(k,x)$ is via the Opdam shift operator. Given a function $m\colon \Sigma \to \mathbf{Z}_+$, Opdam showed in [Op1] that there exists a unique W-invariant differential operator S_m of the form $\delta_m(x)\delta_m(\partial_x)+l.o.t.$, with $\delta_m(x)=\prod_{s\in\Sigma}\alpha_s^{m_s}$ such that

$$L_q S_m = S_m q(\partial)$$

for every $q\in\mathbf{C}[\mathfrak{h}]=\mathbf{C}[q_1,\dots,q_n]$. From this, if we set $\psi(k,x)=S_m e^{(k,x)}$, we get

$$L_q\psi = S_m q(\partial)\, e^{(k,x)} = q(k)\psi\,, \tag{7}$$

$q\in\mathbf{C}[q_1,\dots,q_n]$.

We claim that equation (7) must in fact hold for all $q\in Q_m$. Indeed, near a generic point x, the functions $\psi(wk,x)$ are obviously linearly independent and satisfy (7) for symmetric q. Thus, they are a basis in the space of solutions (we know that this space is $|W|$-dimensional). Consider the matrix of L_q in this basis for any $q\in Q_m$. Since $\psi(k,x)$ is a polynomial multiplied by $e^{(k,x)}$, this matrix must be diagonal with eigenvalues $q(k)$, as desired.

EXAMPLE 3.1. As we have seen in the previous section, for $W=\mathbf{Z}/2$ and $\mathfrak{h}=\mathbf{C}$,

$$S_m=(x\partial-2m+1)(x\partial-2m-1)\cdots(x\partial-1)\,.$$

3.2 BEREST'S FORMULA FOR L_q

We are now going to give an explicit construction of the operators L_q for any $q\in Q_m$.

Let us identify, using our W-invariant scalar product, $\mathfrak{h}$ with $\mathfrak{h}^*$, and let us choose a orthonormal basis $x_1,\dots,x_n$ in $\mathfrak{h}^*$. If $x\in\mathfrak{h}^*$, we will write D_x for the Dunkl operator relative to the vector in $\mathfrak{h}$ corresponding to x under our identification. Thus

$$L=\sum_{i=1}^n D_{x_i}^2\,.$$

PROPOSITION 3.2 (Berest [Be]). *If $q\in Q_m$ is a homogeneous element of degree d, then*

$$(\operatorname{ad} L)^{d+1}q=0\,.$$

Proof. It is enough to prove that

$$((\operatorname{ad} L)^{d+1}q)\psi(k,x)=0\,.$$

Indeed, it follows from the definition of $\psi(k,x)$ that in the ring $\mathcal{D}(U)$ this implies: $((\operatorname{ad} L)^{d+1}q)\,S_m=0$, so that $(\operatorname{ad} L)^{d+1}\,q=0$, since $\mathcal{D}(U)$ is a domain.

Given $q \in Q_m$, we will denote by $L_q^{(k)}$ the operator $q(D_{k_1}, \ldots, D_{k_n})$. Notice that since $\psi(k,x) = \psi(x,k)$, we have $L_q^{(k)}\psi = q(x)\psi$. Thus we deduce, for $p, q, r \in Q_m$,

$$\begin{aligned} L_q r(x) L_p \psi &= L_q r(x) p(k) \psi = p(k) L_q r(x)\, \psi \\ &= p(k) L_q L_r^{(k)} \psi = p(k) L_r^{(k)} L_q \psi = p(k) L_r^{(k)} q(k)\, \psi\,. \end{aligned}$$

It follows that

$$(\operatorname{ad} L)^{d+1} q\psi = (-1)^{d+1} (\operatorname{ad}(\sum_{i=1}^{n} k_i^2))^{d+1} L_q^{(k)} \psi\,.$$

Since L_q is a differential operator of degree d, we get $\operatorname{ad}(\sum_{i=1}^n k_i^2)^{d+1} L_q^{(k)} = 0$, as desired. $\square$

Notice now that the operator $(adL)^d q(x)$ commutes with L. Its symbol is given by $(\operatorname{ad}\Delta)^d q(x) = 2^d d! q(\partial)$. So we deduce the following

COROLLARY 3.3 (Berest's formula, [Be]). *If $q \in Q_m$ is homogeneous of degree d, then*

$$L_q = \frac{1}{2^d d!} (\operatorname{ad} L)^d q(x)\,.$$

Proof. This is clear from Proposition 2.8, once we remark that $(\operatorname{ad} L)^d q(x)$ has the required homogeneity. $\square$

We want to give a representation theoretical interpretation of what we have just seen. Consider the three operators

$$F = \frac{\sum_{i=1}^n x_i^2}{2}\,, \quad E = -\frac{L}{2}\,, \quad H = [E, F]\,. \tag{8}$$

It is easy to check that $[H, E] = 2E$, $[H, F] = -2F$. We deduce that the elements E, F, H span an $\mathfrak{sl}(2)$ Lie subalgebra of $\mathcal{D}(U)$. Thus $\mathfrak{sl}(2)$ acts by conjugation on $\mathcal{D}(U)$. We can then reformulate Proposition 3.2 as follows:

PROPOSITION 3.4. *Any polynomial $q \in Q_m$ of degree d is a lowest weight vector for the $\mathfrak{sl}(2)$-action of weight $-d$ and generates a finite dimensional module (necessarily of dimension $d+1$) for which L_q is a highest weight vector.*

Proof. An easy direct computation shows that

$$H = [E, F] = -\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} + C,$$

where C is a constant. Thus if q is homogeneous of degree d, we have $[H, L_q] = dL_q$.

This and the fact that $[L, L_q] = 0$, implies that L_q is a highest weight vector of weight d. Also since F is a polynomial, we deduce that $\operatorname{ad} F^{d+1} L_q = 0$, so that L_q generates a $(d+1)$-dimensional irreducible $\mathfrak{sl}(2)$-module. □

One last property about these operators is given by

PROPOSITION 3.5 ([FV]). *For any $q \in Q_m$, the operator L_q preserves Q_m.*

Proof. Let us begin by proving that L preserves Q_m.

Take $f \in Q_m$, so that for any $s \in \Sigma$, $f - {}^s f = \alpha_s^{2m_s+1} t$, $t \in \mathbf{C}[\mathfrak{h}]$. Let us start by showing that Lf is a polynomial. Clearly $Lf = \delta_*^{-1} q$, with $q \in \mathbf{C}[\mathfrak{h}]$, and $\delta_* = \prod_{s: m_s \neq 0} \alpha_s$. Since L is W-invariant, $Lf - {}^s(Lf) = L(f - {}^s f)$ is clearly divisible by $\alpha_s^{2m_s-1}$ if $m_s > 0$. In particular, it is always regular along the reflection hyperplane of s. On the other hand, since $Lf - {}^s(Lf) = \delta_*^{-1}(q + {}^s q)$, we deduce that $q + {}^s q$ is divisible by α_s if $m_s > 0$. But then $q = ((q + {}^s q) + (q - {}^s q))/2$ is divisible by α_s if $m_s > 0$, hence it is divisible by δ_*, so that Lf lies in $\mathbf{C}[\mathfrak{h}]$.

We have already remarked that $L(f - {}^s f)$ is divisible by $\alpha_s^{2m_s-1}$ if $m_s > 0$. In fact

$$L(f - {}^s f) = (L\alpha_s^{2m_s+1})t + \alpha_s^{2m_s} \tilde{t},$$

where $\tilde{t}$ is a suitable polynomial.

But since

$$\begin{aligned} L\alpha_s^{2m_s+1} &= 2m_s(2m_s+1)(\alpha_s, \alpha_s)\alpha_s^{2m_s-1} - 2m_{s'}(2m_s+1)\sum_{s' \in \Sigma}(\alpha_{s'}, \alpha_s)\frac{\alpha_s^{2m_s}}{\alpha_{s'}} \\ &= -2m_{s'}(2m_s+1)\sum_{s' \in \Sigma, s' \neq s}(\alpha_{s'}, \alpha_s)\frac{\alpha_s{}^{2m}}{\alpha_{s'}}, \end{aligned}$$

we deduce that $L(f - {}^s f)$ is divisible by $\alpha_s^{2m_s}$. On the other hand, since $L(f - {}^s f) = Lf - {}^s(Lf)$, this polynomial is either zero or it must vanish to odd order on the reflection hyperplane of s. We deduce that it must be divisible by $\alpha_s^{2m_s+1}$, proving that $Lf \in Q_m$.

We now pass to a general L_q, $q \in Q_m$. We may assume that q is homogeneous of, say, degree d. By Corollary 3.3 we have that L_q is a non zero multiple of $(adL)^d(q)$. Since both q and L preserve Q_m, our claim follows. $\square$

3.3 Differential operators on X_m

Now let us return to the algebra of differential operators $\mathcal{D}(X_m)$. Notice that $\mathcal{D}(X_m)$ contains two commutative subalgebras (both isomorphic to Q_m). The first is Q_m itself, the second is the subalgebra $Q_m^\dagger$ consisting of the differential operators of the form L_q with $q \in Q_m$. It is possible to prove

THEOREM 3.6 ([BEG]). *$\mathcal{D}(X_m)$ is generated by Q_m and $Q_m^\dagger$.*

Notice that by Corollary 3.3 we in fact have that $\mathcal{D}(X_m)$ is generated by Q_m and by L.

EXAMPLE 3.7. If $W = \mathbf{Z}/2$, $\mathfrak{h} = \mathbf{C}$ we get that $\mathcal{D}(X_m)$ is generated by the operators

$$x^2, \quad x^{2m+1}, \quad \frac{d^2}{dx^2} - \frac{2m}{x}\frac{d}{dx}.$$

Theorem 3.6 together with Proposition 3.4, imply

COROLLARY 3.8 ([BEG]). *$\mathcal{D}(X_m)$ is locally finite dimensional under the action of the Lie algebra $\mathfrak{sl}(2)$ defined in (8).*

This Corollary implies that our $\mathfrak{sl}(2)$ action on $\mathcal{D}(X_m)$ can be integrated to an action of the group $SL(2)$. In particular we have

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} q = L_q$$

for all $q \in Q_m$. This transformation is a generalization of the Fourier transform, since it reduces to the usual Fourier transform on differential operators on $\mathfrak{h}$ when $m = 0$.

EXAMPLE 3.9. If $W = \mathbf{Z}/2$, $\mathfrak{h} = \mathbf{C}$, we get that the monomials $\{x^{2i}\} \cup \{x^{2i+2m+1}\}$ are (up to constants) all lowest weight vectors for the $\mathfrak{sl}(2)$ action on $\mathcal{D}(X_m)$. x^n has weight $-n$. We deduce that $\mathcal{D}(X_m)$ is isomorphic as a $\mathfrak{sl}(2)$-module to the direct sum of the irreducible representations of dimension $n+1$ for n even or $n = 2(m+i)+1$, each with multiplicity one.

3.4 THE CHEREDNIK ALGEBRA

Let us now return to the algebra $\mathcal{A}$ of operators on U generated by $\mathcal{D}(U)$ and W. This algebra contains the Dunkl operators

$$D_y := \partial_y + \sum_{s\in\Sigma} c_s \frac{(\alpha_s, y)}{\alpha_s}(s-1).$$

LEMMA 3.10. *The following relations hold:*

$$[x_i, x_j] = [D_{x_i}, D_{x_j}] = 0, \quad \forall 1 \le i,j \le n$$
$$[D_{x_i}, x_j] = \delta_{i,j} + \sum_{s\in\Sigma} c_s \frac{(x_i,\alpha_s)(x_j,\alpha_s)}{(\alpha_s,\alpha_s)} s, \quad \forall 1 \le i,j \le n$$
$$wxw^{-1} = w(x), \quad wD_y w^{-1} = D_{w(y)}, \quad \forall w \in W, \, x \in \mathfrak{h}^*, \, y \in \mathfrak{h}.$$

Proof. The proof is an easy computation, except for the relations $[D_{x_i}, D_{x_j}] = 0$, which follow from Theorem 2.6. □

This lemma motivates the following definition.

DEFINITION 3.11 (see e.g. [EG]). The *Cherednik algebra* H_c is an associative algebra with generators $x_i, y_i, i = 1, \dots, n$, and $w \in W$, with defining relations

$$[x_i, x_j] = [y_i, y_j] = 0, \quad \forall 1 \le i,j \le n$$
$$[y_i, x_j] = \delta_{i,j} + \sum_{s\in\Sigma} c_s \frac{(x_i,\alpha_s)(x_j,\alpha_s)}{(\alpha_s,\alpha_s)} s, \quad \forall 1 \le i,j \le n$$
$$wxw^{-1} = w(x), \; wyw^{-1} = w(y), \; w \cdot w' = ww', \; \forall w, w' \in W, \, x \in \mathfrak{h}^*, \, y \in \mathfrak{h}.$$

This algebra was introduced by Cherednik as a rational limit of his double affine Hecke algebra defined in [Ch]. Notice that if $c = 0$ then $H_0 = \mathcal{D}(\mathfrak{h}) \rtimes \mathbf{C}[W]$.

Lemma 3.10 implies that the algebra H_c is equipped with a homomorphism $\phi\colon H_c \to \mathcal{A}$, given by $w \to w$, $x_i \to x_i$, $y_i \to D_{x_i}$.

Cherednik proved the following theorem.

THEOREM 3.12 (Poincaré-Birkhoff-Witt theorem). *The multiplication map*

$$\mu\colon \mathbf{C}[\mathfrak{h}] \otimes \mathbf{C}[\mathfrak{h}^*] \otimes \mathbf{C}[W] \to H_c$$

given by $\mu(f(x) \otimes g(y) \otimes w) = f(x)\, g(y) w$ *is an isomorphism of vector spaces.*

Proof. It is easy to see that the map μ is surjective. Thus, we only have to show that it is injective. In other words, we need to show that monomials $x_1^{i_1} \dots x_n^{i_n} y_1^{j_1} \dots y_n^{j_n} w$ are linearly independent in H_c. To do this, it suffices to show that the images of these monomials under the homomorphism ϕ, i.e. $x_1^{i_1} \dots x_n^{i_n} D_{x_1}^{j_1} \dots D_{x_n}^{j_n} w$, are linearly independent.

Given an element $A \in \mathcal{A}$, writing $A = \sum_{w \in W} P_w w$ with $P_w \in \mathcal{D}(U)$ we define the order of A, ord$A$, as the maximum of the orders of the P_w's. Notice that ord$AB \leq$ord$A$+ord$B$. We now remark that for any sequence of non negative indices $(i_1, \dots, i_n)$,

$$D_{x_1}^{i_1} \cdots D_{x_n}^{i_n} = \partial_{x_1}^{i_1} \cdots \partial_{x_n}^{i_n} + l.o.t.$$

Indeed this is true for D_{x_i}. We proceed by induction on $r = i_1 + \cdots + i_n$. We can clearly assume $i_1 > 0$, so by induction,

$$D_{x_1}^{i_1} \cdots D_{x_n}^{i_n} = (\partial_{x_1} + l.o.t.)(\partial_{x_1}^{i_1 - 1} \cdots \partial_{x_n}^{i_n} + l.o.t.) = \partial_{x_1}^{i_1} \cdots \partial_{x_n}^{i_n} + l.o.t.$$

From this we deduce that for any pair of multiindices $I = (i_1, \dots, i_n)$, $J = (j_1, \dots, j_n)$, $w \in W$, setting $x_I = x_1{}^{i_1} \cdots x_n{}^{i_n}$, $D_J = D_{x_1}^{j_1} \cdots D_{x_n}^{j_n}$, $\partial_J = \partial_{x_1}^{j_1} \cdots \partial_{x_n}^{j_n}$, we have

$$x_I D_J w = x_I \partial_J w + l.o.t.$$

Using this and the linear independence of the elements $x_I \partial_J w$, it is immediate to conclude that the elements $x_I D_J w$ are linearly independent, proving our claim. $\square$

REMARK 1. We see that the homomorphism ϕ identifies H_c with the subalgebra of $\mathcal{A}$ generated by $\mathbf{C}[\mathfrak{h}]$, the Dunkl operators D_y, $y \in \mathfrak{h}$ and W.

REMARK 2. Another way to state the PBW theorem is the following. Let $F^\bullet$ be a filtration on H_c defined by $\deg(x_i) = \deg(y_i) = 1$, $\deg(w) = 0$. Then we have a natural surjective mapping from $\mathbf{C}[\mathfrak{h} \times \mathfrak{h}^*] \rtimes W$ to the associated graded algebra $\mathrm{gr}(H_c)$. The PBW theorem claims that this map is in fact an isomorphism.

3.5 The spherical subalgebra

Let us now introduce the idempotent

$$e = \frac{1}{W} \sum_{w \in W} w \in \mathbf{C}[W].$$

DEFINITION 3.13. The *spherical subalgebra* of H_c is the algebra eH_ce.

Notice that $1 \notin eH_ce$. On the other hand, since $ex = xe = e$ for $x \in eH_ce$, e is the unit for the spherical subalgebra. We can embed both $\mathbf{C}[\mathfrak{h}^*]^W$ and $\mathbf{C}[\mathfrak{h}]^W$ in the spherical subalgebra as follows. Take $f \in \mathbf{C}[\mathfrak{h}^*]^W$ (the other case is identical) and set $m_e(f) = fe$. Since f is invariant, we have $efe = fe^2 = fe = m_e(f)$, so that m_e actually maps $\mathbf{C}[\mathfrak{h}^*]^W$ to eH_ce. The injectivity is clear from the PBW-theorem. As for the fact that m_e is a homomorphism, we have $m_e(fg) = fge = fge^2 = fege = m_e(f)m_e(g)$. From now on, we will consider both $\mathbf{C}[\mathfrak{h}^*]^W$ and $\mathbf{C}[\mathfrak{h}]^W$ as subalgebras of the spherical subalgebra.

3.6 CATEGORY $\mathcal{O}$

We are now going to study representations of the algebras H_c and eH_ce.

DEFINITION 3.14. The category $\mathcal{O}(H_c)$ (resp. $\mathcal{O}(eH_ce)$) is the full subcategory of the category of H_c-modules (resp. eH_ce-modules) whose objects are the modules M such that

1) M is finitely generated.
2) For all $v \in M$, the subspace $\mathbf{C}[\mathfrak{h}^*]^W v \subset M$ is finite dimensional.

We can define a functor

$$F\colon \mathcal{O}(H_c) \to \mathcal{O}(eH_ce)$$

by setting $F(M) = eM$. It is easy to show that $F(M)$ is an object of $\mathcal{O}(eH_ce)$.

We are now going to explain how to construct some modules in $\mathcal{O}(H_c)$ which, by analogy with the case of enveloping algebras of semisimple Lie algebras, we will call Whittaker and Verma modules. First, take $\lambda \in \mathfrak{h}^*$. Denote by $W_\lambda \subset W$ the stabilizer of λ. Take an irreducible W_λ-module τ. We define a structure of $\mathbf{C}[\mathfrak{h}^*] \rtimes \mathbf{C}[W_\lambda]$-module on τ by

$$(fw)v = f(\lambda)(wv) \quad \forall v \in \tau,\ w \in W_\lambda,\ f \in \mathbf{C}[\mathfrak{h}^*].$$

It is easy to see that this action is well defined and we denote this module by $\lambda\#\tau$. We can then consider the H_c-module

$$M(\lambda, \tau) = H_c \otimes_{\mathbf{C}[\mathfrak{h}^*] \rtimes \mathbf{C}[W_\lambda]} \lambda\#\tau.$$

This is called a Whittaker module. In the special case $\lambda = 0$ (and hence $W_\lambda = W$), the module $M(0, \tau)$ is called a Verma module. It is clear that these are objects of $\mathcal{O}$. Notice that as $\mathbf{C}[\mathfrak{h}] \rtimes \mathbf{C}[W]$-module, $M(\lambda, \tau) = \mathbf{C}[\mathfrak{h}] \otimes_{\mathbf{C}} \mathbf{C}[W] \otimes_{\mathbf{C}[W_\lambda]} \tau$.

EXAMPLE 3.15. If $\lambda = 0$ and $\tau = \mathbf{1}$ is the trivial representation of W, the Verma module $M(0, \mathbf{1}) = \mathbf{C}[\mathfrak{h}]$. The action of $\mathbf{C}[\mathfrak{h}]$ is given by multiplication, that of $\mathbf{C}[\mathfrak{h}^*]$ is generated by the Dunkl operators and W acts in the usual way.

3.7 GENERIC c

Opdam and Rouquier have recently studied the structure of the categories $\mathcal{O}(H_c)$, $\mathcal{O}(eH_ce)$, and found that it is especially simple if c is "generic" in a certain sense. Namely, recall that for a W-invariant function $q\colon \Sigma \to \mathbf{C}^*$ one can define the *Hecke algebra* $\mathrm{He}_q(W)$ to be the quotient of the group algebra of the fundamental group of U/W by the relations $(T_s - 1)(T_s + q_s) = 0$, where T_s is the image in U/W of a small half-circle around the hyperplane of s in the counterclockwise direction. It is well known that $\mathrm{He}_q(W)$ is an algebra of dimension $|W|$, which coincides with $\mathbf{C}[W]$ if $q = 1$. It is also known that $\mathrm{He}_q(W)$ is semisimple (and isomorphic to $\mathbf{C}[W]$ as an algebra) unless q_s belongs for some s to a finite set of roots of unity depending on W (see [Hu]).

DEFINITION 3.16. The function c is said to be *generic* if for $q = e^{2\pi i c}$, the Hecke algebra $\mathrm{He}_q(W)$ is semisimple.

In particular, any irrational c is generic, and (more important for us) an integer valued c is generic (since in this case $q = 1$). We can now state the following central result:

THEOREM 3.17 (Opdam-Rouquier [OR]; see also [BEG] for an exposition). *If c is generic (in particular, if c takes non negative integer values), then the irreducible objects in $\mathcal{O}$ are exactly the modules $M(\lambda, \tau)$. Moreover, the category $\mathcal{O}$ is semisimple.*

We also have

THEOREM 3.18 ([OR]). *If c is generic then the functor F is an equivalence of categories.*

From Theorem 3.17 we can deduce

THEOREM 3.19 ([BEG]). *If c is generic, then H_c is a simple algebra.*

In the case $c = 0$, we get the simplicity of $\mathbf{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \rtimes \mathbf{C}[W]$, which is well known.

3.8 The Levasseur-Stafford theorem and its generalization

Let us now recall a result of Levasseur and Stafford:

THEOREM 3.20 ([LS]). *If G is a finite group acting on a finite dimensional vector space V over the complex numbers, then the ring $\mathcal{D}(V)^G$ is generated by the subrings $\mathbf{C}[V]^G$ and $\mathbf{C}[V^*]^G$.*

As an example, notice that if we let $\mathbf{Z}/n\mathbf{Z}$ act on the complex line by multiplication by the n^{th} roots of 1, we deduce that the operator $x\frac{d}{dx}$ can be expressed as a non commutative polynomial in the operators x^n and $\frac{d^n}{dx^n}$, a non-obvious fact. We note also that this theorem has a purely "quantum" nature, i.e. the corresponding "classical" statement, saying that the Poisson algebra $\mathbf{C}[V\times V^*]^G$ is generated, as a Poisson algebra, by $\mathbf{C}[V]^G$ and $\mathbf{C}[V^*]^G$, is in fact false, already for $V=\mathbf{C}$ and $G=\mathbf{Z}/n\mathbf{Z}$.

One can prove a similar result for the algebra eH_ce. Namely, recall that the algebra eH_ce contains the subalgebras $\mathbf{C}[\mathfrak{h}]^W$, and $\mathbf{C}[\mathfrak{h}^*]^W$.

THEOREM 3.21 ([BEG]). *If c is generic then the two subalgebras $\mathbf{C}[\mathfrak{h}]^W$ and $\mathbf{C}[\mathfrak{h}^*]^W$ generate eH_ce.*

Notice that if $c=0$, then $eH_0e=\mathcal{D}(\mathfrak{h})^W$, so Theorem 3.21 reduces to the Levasseur-Stafford theorem.

REMARK. It is believed that this result holds without the assumption of generic c. Moreover, it is known to be true for all c if W is a Weyl group not of type E and F, since in this case Wallach proved that the corresponding classical statement for Poisson algebras holds true. Nevertheless, the genericity assumption is needed for the proof, because, similarly to the proof of the Levasseur-Stafford theorem, it is based on the simplicity of H_c.

3.9 The action of the Cherednik algebra to quasi-invariants

We now go back to the study of Q_m. Notice that the algebra eH_me acts on $\mathbf{C}[\mathfrak{h}]^W$, since e gives the W-equivariant projection of $\mathbf{C}[\mathfrak{h}]$ onto $\mathbf{C}[\mathfrak{h}]^W$. It is clear that this action is by differential operators. For instance, the subalgebra $\mathbf{C}[\mathfrak{h}]^W\subset eH_me$ acts by multiplication. Also, an element $q\in\mathbf{C}[\mathfrak{h}^*]^W\subset eH_me$ acts via the operator $q(D_{x_1},\dots,D_{x_n})$. By definition this operator coincides with L_q on $\mathbf{C}[\mathfrak{h}]^W$.

The following important theorem shows that this action extends to Q_m.

THEOREM 3.22 ([BEG]). *There exists a unique representation of the algebra eH_me on Q_m in which an element $q \in \mathbf{C}[\mathfrak{h}]^W$ acts by multiplication and an element $q \in \mathbf{C}[\mathfrak{h}^*]^W$ by L_q.*

Proof. Since by Proposition 3.5, L_q preserves Q_m, we get a uniquely defined representation of the subalgebra of eH_me generated by $\mathbf{C}[\mathfrak{h}]^W$ and $\mathbf{C}[\mathfrak{h}^*]^W$ on Q_m. The result now follows from Theorem 3.21. $\square$

3.10 PROOF OF THEOREM 1.8

Finally we can prove Theorem 1.8.

To do this, observe that as an eH_me-module, Q_m is in the category $\mathcal{O}(eH_me)$, and $\mathbf{C}[\mathfrak{h}^*]^W$ acts locally nilpotently in Q_m (by degree arguments). We can now apply Theorem 3.18 and Theorem 3.17 and deduce that Q_m is a direct sum of modules of the form $eM(0,\tau)$. As a $\mathbf{C}[\mathfrak{h}] \rtimes \mathbf{C}[W]$-module, $M(0,\tau) = \mathbf{C}[\mathfrak{h}] \otimes \tau$. On the other hand, by Chevalley's theorem, there is an isomorphism $\mathbf{C}[\mathfrak{h}] \simeq \mathbf{C}[\mathfrak{h}]^W \otimes \mathbf{C}[W]$, commuting with the action of W and $\mathbf{C}[\mathfrak{h}]^W$. Thus we get an isomorphisms of $\mathbf{C}[\mathfrak{h}]^W$-modules

$$eM(0,\tau) \simeq (M(0,\tau))^W \simeq \mathbf{C}[\mathfrak{h}]^W \otimes (\mathbf{C}[W] \otimes \tau)^W \simeq \mathbf{C}[\mathfrak{h}]^W \otimes \tau\,,$$

proving that $eM(0,\tau)$ and hence Q_m is a free $\mathbf{C}[\mathfrak{h}]^W$-module. $\square$

EXAMPLE 3.23. For $W = \mathbf{Z}/2$ and $\mathfrak{h} = \mathbf{C}$, take the polynomials 1, x^{2m+1}. Notice that $L(1) = L(x^{2m+1}) = 0$ while $s(1) = 1$, $s(x^{2m+1}) = -x^{2m+1}$, $s \in \mathbf{Z}/2$ being the element of order two. It follows that Q_m as a eH_me-module is the direct sum of $\mathbf{C}[x^2] \oplus x^{2m+1}\mathbf{C}[x^2]$. These modules are irreducible. Moreover, $\mathbf{C}[x^2] \simeq eM(0,\mathbf{1})$, $x^{2m+1}\mathbf{C}[x^2] \simeq eM(0,\varepsilon)$, ε being the sign representation.

3.11 PROOF OF THEOREM 1.15

Let I be a nonzero two-sided ideal in $\mathcal{D}(X_m)$. First we claim that I nontrivially intersects Q_m. Indeed, otherwise let $K \in I$ be a lowest order nonzero element in I. Since the order of K is positive, there exists $f \in Q_m$ such that $[K,f] \neq 0$. Then $[K,f] \in I$ is of smaller order than K, a contradiction.

Now let $f \in Q_m$ be an element of I. Then $g = \prod_{w \in W} {}^w f \in I$. But g is W-invariant. This shows that the intersection J of I with the subalgebra H_m in $\mathcal{D}(X_m)$ is nonzero. But H_m is simple by Theorem 3.19, so $J = H_m$. Hence, $1 \in J \subset I$, and $I = \mathcal{D}(X_m)$. $\square$