4. Main theorem

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **49 (2003)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **26.09.2024**

That is, $\sigma_\epsilon(x) = \sigma_\epsilon(y)$. This holds for any $\epsilon > 0$. Since $(\sigma_t)_{t\in\mathbf{R}^+}$ depends continuously on t, we have $\sigma_0(x) = \sigma_0(y)$, whence $x = y$. We have thus proved that $d_{(\widetilde{X},\mathcal{H})}$ does not vanish outside the diagonal of $\widetilde{X} \times \widetilde{X}$. The conclusion that this is a distance is now straightforward.

By definition of the telescopic distance, for any x, y in $\widetilde{X}$, for any $\epsilon > 0$, there exists a telescopic path p between x and y such that $|p|_{(\widetilde{X},\mathcal{H})} \leq d_{(\widetilde{X},\mathcal{H})}(x,y)+\epsilon$. We choose $\epsilon < \min(d_{(\widetilde{X},\mathcal{H})}(x,y), 1)$. We consider the maximal collection of points $x_0, \ldots, x_k$ in p such that $x_0 = i(p)$, $x_k = t(p)$, and that the telescopic length of the subpath p_i of p between x_{i-1} and x_i is equal to ϵ for $i = 1, \ldots, k-1$. The maximality of the collection $\{x_0, x_1, \ldots, x_k\}$ implies that the telescopic length of the subpath p_k of p between x_{k-1} and x_k is at most ϵ. By definition $d_{(\widetilde{X},\mathcal{H})}(x_{i-1}, x_i) \leq |p_i|_{(\widetilde{X},\mathcal{H})}$ for $i = 1, \ldots, k$. Thus $d_{(\widetilde{X},\mathcal{H})}(x_{i-1}, x_i) \leq 1$ for any $i = 1, \ldots, k$ and $\sum_{i=1}^{k} d_{(\widetilde{X},\mathcal{H})}(x_{i-1}, x_i) \leq |p|_{(\widetilde{X},\mathcal{H})}$. The choice of $\epsilon < d_{(\widetilde{X},\mathcal{H})}(x,y)$ then implies that $\sum_{i=1}^{k} d_{(\widetilde{X},\mathcal{H})}(x_{i-1}, x_i) \leq 2d_{(\widetilde{X},\mathcal{H})}(x,y)$. Therefore $x_0, x_1, \ldots, x_k$ is a $(1,2)$-quasi geodesic chain between x and y. □

REMARK 3.7. In nice cases, for instance in the case where the forest-stack is a proper metric space, the forest-stack is a true geodesic space.

4. MAIN THEOREM

DEFINITION 4.1. Let $(\widetilde{X}, f, \sigma_t, \mathcal{H})$ be a forest-stack equipped with some horizontal metric $\mathcal{H}$.

1. The semi-flow is a *bounded-cancellation semi-flow* (with respect to $\mathcal{H}$) if there exist $\lambda_- \geq 1$ and $K \geq 0$ such that for any real $r \in \mathbf{R}$, for any horizontal geodesic $g \in f^{-1}(r)$, for any $t \geq 0$, $|[g]_{r+t}|_{r+t} \geq \lambda_-^{-t}|g|_r - K$.
2. The semi-flow is a *bounded-dilatation semi-flow* (with respect to $\mathcal{H}$) if there exists $\lambda_+ \geq 1$ such that for any real $r \in \mathbf{R}$, for any horizontal geodesic $g \in f^{-1}(r)$, for any $t \geq 0$, $|[g]_{r+t}|_{r+t} \leq \lambda_+^{t}|g|_r$.

REMARK 4.2. The reader can observe a dissymetry between the bounded-cancellation and bounded-dilatation properties, in the sense that the latter does not allow any additive constant. This is really necessary, since several proofs fail (e.g. those of Propositions 8.1 or 9.1) if an additive constant is allowed here.

DEFINITION 4.3. Let $(\widetilde{X}, f, \sigma_t, \mathcal{H})$ be a forest-stack equipped with some horizontal metric $\mathcal{H}$.

1. The semi-flow is *hyperbolic* (with respect to $\mathcal{H}$) if it is a bounded-dilatation and bounded-cancellation semi-flow with respect to $\mathcal{H}$ and there exist $\lambda > 1$, t_0, $M \geq 0$ such that, for any horizontal geodesic $g \in f^{-1}(r)$ with $|g|_r \geq M$, either

- $|[g]_{r+nt_0}|_{r+nt_0} \geq \lambda^{nt_0}|g|_r$ for any integer $n \geq 1$, or
- for any integer $n \geq 1$, some geodesic preimage g_{-nt_0} of g satisfies $|g_{-nt_0}|_{r-nt_0} \geq \lambda^{nt_0}|g|_r$.

2. The semi-flow is *strongly hyperbolic* (with respect to $\mathcal{H}$) if it is hyperbolic and also satisfies the following condition:

Any horizontal geodesic $g \in f^{-1}(r)$ with $|g|_r \geq M$, which admits geodesic preimages in distinct connected components of the stratum $f^{-1}(r-\epsilon)$ for arbitrarily small $\epsilon > 0$, admits a preimage g_{-nt_0} in each connected component of the stratum $f^{-1}(r-nt_0)$ such that $|g_{-nt_0}|_{r-nt_0} \geq \lambda^{nt_0}|g|_r$.

Let us observe that if the strata are connected, then a hyperbolic semi-flow is strongly hyperbolic.

We can now state the main theorem of this paper.

THEOREM 4.4. *Let $(\widetilde{X}, f, \sigma_t, \mathcal{H})$ be a connected forest-stack. If $(\sigma_t)_{t\in\mathbf{R}^+}$ is strongly hyperbolic with respect to $\mathcal{H}$ then $\widetilde{X}$ is a Gromov-hyperbolic metric space for any telescopic metric associated to $\mathcal{H}$.*

At this point, the reader might prefer to read Sections 12 and 13, which give applications, and so illustrations, of this theorem to the cases of mapping-telescope spaces and of mapping-torus groups.

REMARK 4.5 (About the necessity of the bounded-cancellation property). We observe that the Cayley complex of a Baumslag-Solitar group $BS(1,m) = \langle a, b\ ;\ b^{-1}ab = a^m \rangle$ is a forest-stack with a hyperbolic semi-flow. But this is not a Gromov hyperbolic 2-complex with respect to the telescopic metric. What happens here is that the semi-flow is hyperbolic but not strongly hyperbolic.

An example of a non Gromov-hyperbolic locally finite forest-stack with connected strata and a semi-flow satisfying all the desired properties, with the exception of the bounded-cancellation property (first item of Definition 4.1) is constructed as follows. We start with the forest-stack $\mathcal{R} = (\mathbf{R}^2, f, \sigma_t, \mathcal{H})$ defined in Section 1 and equipped with the associated telescopic metric. We consider copies $\mathcal{R}_i$, $i = 0, 1, 2, \ldots$ of $\mathcal{R}$. We glue them to $\mathcal{R}$ as illustrated in Figure 2, that is by creating an infinite sequence of pockets of increasing size.

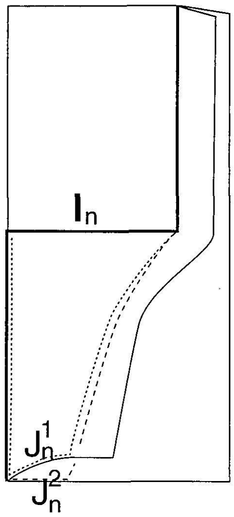

FIGURE 2
(A pocket)

We now attach copies of the negative half-plane of $\mathcal{R}$, along the horizontal lines with integer y-coordinate of the copies $\mathcal{R}_i$ of $\mathcal{R}$ considered above. In order to get a forest-stack whose strata are trees, we now identify a vertical half-line in each of the copies of the negative half-plane, ending at the horizontal line along which this copy was glued, to the corresponding vertical half-line in $\mathcal{R}$. In this way, we get a forest-stack whose strata are trees and whose semi-flow is as anounced. This forest-stack is not Gromov-hyperbolic because in each pocket (see Figure 2) the horizontal interval I_n admits two preimages J_n^1, J_n^2 so that there are two telescopic geodesics joining the endpoints of I_n. These are the concatenation of J_n^1 and J_n^2 with the two vertical segments joining their endpoints to the endpoints of I_n. Since, by construction, there are pockets of arbitrarily large size, these two telescopic geodesics can be arbitrarily far from one another, so that the forest-stack is not Gromov-hyperbolic.

5. PRELIMINARY WORK

We consider a forest-stack $(\widetilde{X}, f, \sigma_t, \mathcal{H})$ equipped with a horizontal metric $\mathcal{H}$ such that the semi-flow $(\sigma_t)_{t\in\mathbf{R}^+}$ is strongly hyperbolic. Definition 4.3 introduces three *constants of hyperbolicity*, denoted by λ, t_0, M in the