# 2. Mapping-telescopes and forest-stacks

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **49 (2003)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **21.09.2024**

REMARK 1.1. The above computations fail, and the space is no longer Gromov-hyperbolic, if one replaces $d_y = \lambda^{|y|}d_0$ by $d_y = P(|y|)d_0$, where $P(.)$ is a polynomial function of $y$. Indeed, in this case, the length of the horizontal interval between the two considered orbits, evaluated at the height where the minimum of the length-function $f(t)$ is attained, depends, even in the optimal case, on the horizontal length of the interval connecting one point to the orbit of the other. Whereas in the exponential case it equals $\frac{2}{\ln \lambda}$ unless it belongs to the horizontal axis.

## 2. MAPPING-TELESCOPES AND FOREST-STACKS

Let $X$ be a topological space. Call $X$ a *topological tree* if there exists a unique arc between any two points in $X$. A *topological forest* is a union of disjoint topological trees. By 'arc' we mean the image of an injective path. A path in $X$ is a continuous map from a bounded interval of the real line into $X$. A *forest-map* is a continuous map of a topological forest into itself.

DEFINITION 2.1. Let $\psi\colon X \to X$ be a forest-map. The *mapping-telescope* $K_\psi$ *of* $(\psi, X)$ is the topological space resulting from $K_X = \bigsqcup_{n\in\mathbf{Z}} X \times [n, n+1]$ by the identification of each point $(x, n+1) \in X \times [n, n+1]$ with the point $(\psi(x), n+1) \in X \times [n+1, n+2]$.

Let us examine somewhat more closely the topology of these mapping-telescopes.

For any integer $n \in \mathbf{Z}$, for any $(x, r) \in X \times [n, n+1]$, for any real number $t \geq 0$, we define $\tilde{\sigma}_t((x,r))$ as the point $(\psi^{E[t-(n+1-r)]+1}(x), r+t)$ in $X \times [E[r+t], E[r+t]+1]$, where $E[r]$ denotes the integer part of $r$. The map $\tilde{\sigma}_t$ is defined on $K_X$ (the disjoint union of the $X \times [n, n+1]$) for every $t \geq 0$. Moreover $\tilde{\sigma}_{t+t'} = \tilde{\sigma}_t \circ \tilde{\sigma}_{t'}$.

If $a = (x, n+1) \in X \times [n+1, n+2]$, then $\tilde{\sigma}_t(a) = (\psi^{E[t]}(x), n+1+t) \in [n+1+E[t], E[t]+n+2]$. Whereas if $a = (x, n+1) \in X \times [n, n+1]$ then $\tilde{\sigma}_t(a) = (\psi^{E[t]+1}(x), n+1+t) \in X \times [n+1+E[t], E[t]+n+2]$, which is equal to $\tilde{\sigma}_t(b)$ with $b = (\psi(x), n+1) \in X \times [n+1, n+2]$. Therefore $(\tilde{\sigma}_t)_{t\in\mathbf{R}^+}$ descends to the mapping-telescope $K_\psi$, where it defines a one parameter family $(\sigma_t)_{t\in\mathbf{R}^+}$ of continuous maps of $K_\psi$. This family depends continuously on the parameter $t \in \mathbf{R}^+$. It satisfies furthermore $\sigma_0 = \mathrm{Id}_{K_\psi}$ and $\sigma_{t+t'} = \sigma_t \circ \sigma_{t'}$. Such a family is called a *semi-flow* on $K_\psi$.

Let $f\colon K_\psi \to \mathbf{R}$ be defined by $f(a) = r$ if $a \in X \times \{r\}$. Then $f$ is a continuous surjective map. The preimage of any real number $r$ is $X \times \{r\}$, a topological forest. Furthermore, for any $t \geq 0$, $f \circ \sigma_t = \tau_t \circ f$, where $\tau_t\colon \mathbf{R} \to \mathbf{R}$ is defined by $\tau_t(r) = r + t$.

We extracted above the two properties shared by mapping-telescopes which are really important for our work. We now define a class of spaces which satisfy these two properties, and in particular generalize the mapping-telescopes.

DEFINITION 2.2. Let $X$ be a topological space. Let $(\sigma_t)_{t\in\mathbf{R}^+}$ be a semi-flow on $X$. Let $f\colon X \to \mathbf{R}$ be a surjective continuous map such that:

1. For any real number $r$, the *stratum* $f^{-1}(r)$ is a topological forest.
2. For any $t \geq 0$, $f \circ \sigma_t = \tau_t \circ f$, where $\tau_t(r) = r + t$ for any real number $r$.

Then $X$ is a *forest-stack*, denoted by $(X, f, \sigma_t)$.

REMARK 2.3. All the strata of a mapping-telescope are homeomorphic. This is not required in the definition of a forest-stack.

As we just saw, a mapping-telescope is an example of a forest-stack. In Section 13, we show that a Cayley complex for the mapping-torus group of an injective free group endomorphism is a mapping-telescope of a forest-map, and thus a forest-stack. The reader can also find there, and in Section 12, an illustration of the horizontal and vertical metrics on forest-stacks, which we are now going to define.

## 3. METRICS

The aim of this section is to introduce a particular metric on forest-stacks, called the *telescopic metric*. We sometimes deal with metric spaces which are not necessarily connected, for instance forests. In this case, when considering the distance between two points, it will always be tacitly assumed that the two points lie in a same connected component of the space.

### 3.1 HORIZONTAL AND VERTICAL METRICS

Let us consider a forest-stack $(\widetilde{X}, f, \sigma_t)$, see Definition 2.2. We want to define a natural metric on the orbits of the semi-flow.