# §6. Mean curvature

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## §6. Mean curvature

Let $X$ be a closed $n$-dimensional manifold with a Riemannian metric $g$. Suppose that iterated graphs $\Gamma_k \subset X^k$ are smooth of dimension $n$. Denote by $\mathrm{Cu}(\gamma)$, $\gamma \in \Gamma_k$, the absolute value of the mean curvature of $\Gamma_k$ at $\gamma$. Set

$$\mathrm{lome}_g \Gamma = \limsup_{k\to\infty} \frac{1}{k} \log\Big(1 + \int_{\Gamma_k} [\mathrm{Cu}(\gamma)]^n d\gamma\Big) .$$

When $\Gamma_k$ are minimal and $\mathrm{lome}_g = 0$ we know that $h \leq$ "lov".

More generally,

$$h(\gamma) \leq \mathrm{lov}\, \Gamma + \mathrm{lome}_g \Gamma . \tag{6.0}$$

*Proof.* Despite the possible dependence of "lome" upon the choice of $g$, we can proceed as before and reduce (6.0) to the following local estimate:

Take $V$ in the Euclidean space $\mathbf{R}^{\ell=kn}$ and suppose its boundary does not intersect the ball $B_{2\epsilon}$ centered at $v_0 \in V$. Then

$$\epsilon^{-n} \mathrm{Vol}\ V + \int_V Cu^n(v)dv \geq C_1 \ell^{C_2} , \tag{6.1}$$

where $C_1$ and $C_2$ are constants depending only on $n$.

To prove (6.1) we consider the normal bundle $N$ of $V$ and its canonical map $F$ into $\mathbf{R}^\ell$. The Jacobian $J$ of $F$ at a point $v + \nu t$ (where $v \in V$, and $\nu$ is the unit vector at $v$ normal to $V$) is equal to $\prod_{i=1}^n (1 + k_i t)$, where $k_i$ are the principal curvatures in the direction $\nu$.

If the distance from $v+\nu t$ to $V$ is equal to $t$, then $1+k_i t \geq 0$, $i = 1, \ldots, h$, and so

$$J \leq A_n\big(1 + t^n Cu^n(v)\big) . \tag{6.2}$$

Now we observe that every point of the ball $B_\epsilon$ can be joined by a shortest normal with $V$ and so

$$C_\ell \epsilon^\ell = \mathrm{Vol}\ B_\epsilon \leq A_n C_{\ell-n} \epsilon^{\ell-n} \int_V \big(1 + \epsilon^n Cu^n(v)\big)\, dv ,$$

where $C_\ell$ and $C_{\ell-n}$ are volumes of unit balls in $\mathbf{R}^\ell$ and $\mathbf{R}^{\ell-n}$. The last inequality implies (6.1) and so (6.0) is proved.

The inequality (6.2) was extended by Karcher and Heinze to general Riemannian manifolds. Discussions with Karcher about such inequalities influenced my reasoning in this section.