7. The -cone of $S^{2n} \times S^{2m}$ and the positive cone of $S^2 \times S^{2n}$

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We now turn to the product $S^{2n-1} \times S^{2m-1}$. From the six-term exact sequence of the pair $(S^{2n-1} \times S^{2m-1}, S^{2n-1} \vee S^{2m-1})$, with quotient the smash product $S^{2n-1} \wedge S^{2m-1}$ homeomorphic to $S^{2m+2n-2}$, we get an isomorphism

$$q^* : \widetilde{K}(S^{2m+2n-2}) \longrightarrow \widetilde{K}(S^{2n-1} \times S^{2m-1})$$

induced by the quotient map $q: S^{2n-1} \times S^{2m-1} \longrightarrow S^{2m+2n-2}$. By Theorem 4.1, the space $Y = S^{2n+2m-2}$ satisfies the hypothesis of Proposition 5.5 and we deduce the

THEOREM 6.2. *The map $q: S^{2n-1} \times S^{2m-1} \longrightarrow S^{2m+2n-2}$ induces an isomorphism of positive cones, and, for $S^{2n-1} \times S^{2m-1}$, the γ-cone and the c-cone coincide with the positive cone:*

$$K_+(S^{2m+2n-2}) \overset{q^*}{\cong} K_+(S^{2n-1} \times S^{2m-1}) = K_\gamma(S^{2n-1} \times S^{2m-1}).$$

REMARK 6.3. According to Blackadar ([Bla2], 6.10.2), the positive cone of the n-torus $(S^1)^n$ has been partially computed by Villadsen.

7. THE γ-CONE OF $S^{2n} \times S^{2m}$ AND THE POSITIVE CONE OF $S^2 \times S^{2n}$

The positive cone was rather easy to compute for a product of an odd-dimensional sphere by any sphere, whereas the case of a product of two even-dimensional spheres is much more involved. On the other hand, the γ-cone of such a product is in the scope of the present notes. We perform this calculation by computing the c-cone and appealing to Proposition 3.3.

By the Künneth theorem, we have an isomorphism

$$K(S^{2n}) \otimes K(S^{2m}) \longrightarrow K(S^{2n} \times S^{2m}), \; \xi \otimes \eta \longmapsto p^*(\xi) \cdot q^*(\eta),$$

where p and q are the projections onto the factors. Writing $\widetilde{K}(S^{2n}) = \mathbf{Z} \cdot x_1$ and $\widetilde{K}(S^{2m}) = \mathbf{Z} \cdot x_2$, and letting $y_1 := p^*(x_1)$ and $y_2 := q^*(x_2)$, we deduce that

$$\widetilde{K}(S^{2n} \times S^{2m}) = \mathbf{Z} \cdot y_1 \oplus \mathbf{Z} \cdot y_2 \oplus \mathbf{Z} \cdot y_1 y_2 .$$

The product structure on $\widetilde{K}(S^{2n} \times S^{2m})$ is given by $y_1^2 = 0$ and $y_2^2 = 0$. One has $y_1 y_2 = \pi^*(y)$, where $\pi: S^{2n} \times S^{2m} \longrightarrow S^{2n} \wedge S^{2m} \cong S^{2n+2m}$ and y is a suitable generator of $\widetilde{K}(S^{2n+2m})$. Let $i: S^{2n} \hookrightarrow S^{2n} \times S^{2m}$ and $j: S^{2m} \hookrightarrow S^{2n} \times S^{2m}$ be the inclusions. One has $i^*(y_1) = x_1$ and $j^*(y_2) = x_2$, and (by Theorem 4.1 and a double application of Proposition 5.1), for any $k \in \mathbf{Z} \setminus \{0\}$, one has g-dim$(ky_1)$ = g-dim$(kx_1) = n$; similarly g-dim(ky_2) = g-dim$(kx_2) = m$. This justifies that, from now on, we write x_1 and x_2 for y_1 and y_2 respectively.

Let $a_1 \in H^{2n}(S^{2n}; \mathbf{Z})$ and $a_2 \in H^{2m}(S^{2m}; \mathbf{Z})$ be suitable generators (referring to Proposition 2.4). As before, it is justified to write

$$\widetilde{H}^*(S^{2n} \times S^{2m}; \mathbf{Z}) = \mathbf{Z} \cdot a_1 \oplus \mathbf{Z} \cdot a_2 \oplus \mathbf{Z} \cdot a_1 a_2 .$$

Let us assume $n \leq m$. Consider an element $x = ax_1 + bx_2 + lx_1x_2$ in the group $\widetilde{K}(S^{2n} \times S^{2m})$. For the Chern classes, invoking Proposition 2.4 and "exponentiality" of the total Chern class, we compute

$$\begin{aligned} c(x) &= c(ax_1)c(bx_2)c(lx_1x_2) \\ &= 1 + (-1)^{n-1}a(n-1)! \cdot a_1 + (-1)^{m-1}b(m-1)! \cdot a_2 \\ &\qquad + (-1)^{n+m}(ab(n-1)!\,(m-1)! - l(n+m-1)!) \cdot a_1a_2 . \end{aligned}$$

This immediately gives the γ-cone (which coincides with the c-cone) in terms of the γ-dimension function.

THEOREM 7.1. *For $n \leq m$, the γ-dimension on $\widetilde{K}(S^{2n} \times S^{2m})$ is given as follows: for $x = ax_1 + bx_2 + lx_1x_2 \in \widetilde{K}(S^{2n} \times S^{2m})$, one has*

$$\gamma\text{-dim}(x) = \begin{cases} 0 & \text{if } a = b = l = 0 \\ n & \text{if } a \neq 0,\ b = l = 0 \\ m & \text{if } b \neq 0,\ l = ab\,(n-1)!\,(m-1)!/(n+m-1)! \\ n+m & \text{if } l \neq ab\,(n-1)!\,(m-1)!/(n+m-1)! \end{cases}$$

Moreover, for $k \neq 0$, one has

$$\text{g-dim}(kx_1) = n \quad \textit{and} \quad \text{g-dim}(kx_2) = m .$$

This theorem allows us to give some interesting information on the positive cone of the product $S^{2n} \times S^{2m}$. We will state the result as Theorem 8.2 in the following section, because the tools developed there allow one to make a crucial improvement.

Combined with Theorem 2.3, Theorem 7.1 enables one to compute completely the positive cone of $S^2 \times S^{2n}$.

THEOREM 7.2. *For the product $S^2 \times S^{2n}$, we have*

$$K_+(S^2 \times S^{2n}) = K_c(S^2 \times S^{2n}) = K_\gamma(S^2 \times S^{2n}) .$$

The latter is given by Theorem 7.1.