3. The -cone and the c-cone

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computation (based on Proposition 2.4) for the sphere S^{2n} shows that one has $q_n = (-1)^{n-1}(n-1)!$, as claimed. □

3. The γ-cone and the c-cone

In general, the problem of computing the geometric dimension of vector bundles is very complicated, as is any general lifting problem in homotopy theory. So, the same is true for the positive cone. That is why we now introduce what we call the γ-cone and the c-cone. They are supposed to be easier to compute and might be good approximations to the positive cone. As we will see, these two cones coincide for torsion-free spaces.

DEFINITION 3.1.
i) The *γ-cone* of X is defined by

$$K_\gamma(X) := \{(n, x) \in \mathbf{Z} \oplus \widetilde{K}(X) \mid \gamma^k(x) = 0 \text{ for all } k > n\}.$$

The *γ-dimension* of a class $x \in \widetilde{K}(X)$, denoted by $\gamma\text{-dim}(x)$, is the least integer n such that $\gamma^k(x) = 0$ for all $k > n$, in other words, it is the degree (in the variable t) of the polynomial $\gamma_t(x)$.

ii) The *c-cone* of X is defined by

$$K_c(X) := \{(n, x) \in \mathbf{Z} \oplus \widetilde{K}(X) \mid c_k(x) = 0 \text{ for all } k > n\}.$$

The *c-dimension* of a class $x \in \widetilde{K}(X)$, denoted by $\text{c-dim}(x)$, is the least integer n such that $c_k(x) = 0$ for all $k > n$, in other words, it is the degree (in the variable t) of the polynomial $c_x(t)$.

Let us point out that the "lower boundary" of the positive cone $K_+(X)$, as a subset of $\widetilde{K}(X) \oplus \mathbf{Z}$, coincides with the graph of the geometric dimension function $\text{g-dim}\colon \widetilde{K}(X) \longrightarrow \mathbf{Z}$ (the positive elements consisting exactly of the boundary and the points located above it). The analogous statements hold for the γ-cone and the c-cone with respect to the corresponding dimension function.

The following results on these objects follow readily from our preliminaries on K-theory.

PROPOSITION 3.2. *Let X be a connected finite CW-complex. Then*

i) $\text{g-dim}(x) \leq \dim(X)/2$, *for any* $x \in \widetilde{K}(X)$;

ii) $\gamma\text{-dim}(x) \leq \text{g-dim}(x)$, *for any* $x \in \widetilde{K}(X)$;

iii) $K_+(X) \subseteq K_\gamma(X)$;

iv) $\text{c-dim}(x) \leq \text{g-dim}(x)$, *for any* $x \in \widetilde{K}(X)$;

v) $K_+(X) \subseteq K_c(X)$.

This proposition shows that the γ-cone and the c-cone are approximations of the positive cone, more precisely, that they constitute upper bounds of the latter.

It turns out that the γ-cone and the c-cone coincide for torsion-free spaces, i.e. those spaces having no torsion in their integral cohomology.

PROPOSITION 3.3. *Let X be a connected finite CW-complex. If X is torsion-free, then*

$$K_\gamma(X) = K_c(X).$$

Proof. The result follows immediately from Proposition 2.2 and injectivity of the Chern character for a torsion-free space. □

It is worth mentioning that there is no general comparison statement for the γ-cone and the c-cone, i.e. there are spaces with torsion for which the γ-cone is not contained in the c-cone, and spaces with torsion for which the c-cone is not contained in the γ-cone. Moreover, there exist spaces for which the γ-cone and the c-cone strictly contain the positive cone (the product $S^4 \times S^4$ is such an example as we will later see). We now illustrate the situation by three examples.

EXAMPLES.

i) Let $j\colon BSU(3) \longrightarrow BU(3)$ be the map induced by the inclusion of the special unitary group $SU(3)$ in $U(3)$. Then the composition map

$$BSU(3) \xrightarrow{\ j\ } BU(3) \xrightarrow{\ \gamma_3\ } BU$$

lifts to a map $f\colon BSU(3) \longrightarrow BSU$. Consider W the homotopy fibre of f. It enters in a pull-back diagram

$$\begin{array}{ccc} SU & = & SU \\ \cap\downarrow & & \cap\downarrow \\ W & \longrightarrow & PBSU \\ \pi\downarrow & & \downarrow \\ BSU(3) & \xrightarrow{f} & BSU \end{array}$$

where $SU \simeq \Omega BSU \hookrightarrow PBSU \to BSU$ is the path-loop fibration of BSU. The Leray-Serre spectral sequence in cohomology for this fibration is well-known and maps via f^* to the corresponding spectral sequence for the fibration π. By Lemma 2.5, one has

$$f^*(\widetilde{c}_3) = j^* \circ \widetilde{\gamma}_3^*(c_3(\widetilde{\rho}_3)) = c_3(\widetilde{\gamma}_3) = 2\widetilde{c}_3 \,.$$

Similarly, one has $f^*(\widetilde{c}_2) = c_2(\widetilde{\gamma}_3)$, which is easily seen to vanish. For the cohomology of W in degree ≤ 6, letting $a_4 := \pi^*(\widetilde{c}_2)$ and $b_6 := \pi^*(\widetilde{c}_3)$, we have computed that $x_3^2 = 0$ and

$$H^{\leq 6}(W;\, \mathbf{Z}) = \mathbf{Z}\cdot 1 \oplus \mathbf{Z}\cdot x_3 \oplus \mathbf{Z}\cdot a_4 \oplus \mathbf{Z}\cdot x_5 \oplus \underbrace{\mathbf{Z}\cdot b_6}_{\cong \mathbf{Z}/2} \cong \mathbf{Z}^4 \oplus \mathbf{Z}/2 \,,$$

where $\deg(x_{2j+1}) = 2j+1$. The inclusion $i\colon Y := W^{[7]} \hookrightarrow W$ of the 7-skeleton of W induces an isomorphism in cohomology up to degree 6. If we let $x := i^* \circ \pi^* \circ j^*(\widetilde{\rho}_3) \in \widetilde{K}(Y)$, we find $c_3(x) = b_6 \neq 0$, whereas $\gamma^k(x) = 0$, for all $k \geq 3$. Indeed, this is clear for $k \geq 4$ since then $\gamma^k(\widetilde{\rho}_3) = 0$, and $\gamma^3(x) = 0$ because its classifying map is the composition $f \circ \pi \circ i$, which is homotopically trivial. Thus c-dim$(x) = 3$ and γ-dim$(x) \leq 2$. Consequently, Y is a connected finite CW-complex with a strict inclusion

$$K_c(Y) \subsetneq K_\gamma(Y) \,.$$

ii) Consider the Moore space $M = M(\mathbf{Z}/2,\, 5)$, i.e. the mapping cone of a continuous map $f\colon S^5 \xrightarrow{2} S^5$ of degree two, or more explicitly, $M = C_f = S^5 \cup_2 e^6$. The exact sequences in cohomology and in K-theory of the cofibration $S^5 \hookrightarrow M \overset{q}{\twoheadrightarrow} M/S^5 \simeq S^6$ give epimorphisms

$$q^* : \mathbf{Z} \cong H^6(S^6;\, \mathbf{Z}) \twoheadrightarrow H^6(M;\, \mathbf{Z}) \cong \mathbf{Z}/2$$

$$q^* : \mathbf{Z} \cong \widetilde{K}(S^6) \twoheadrightarrow \widetilde{K}(M) \cong \mathbf{Z}/2 \,.$$

Let x and a be suitable generators of $\widetilde{K}(S^6)$ and of $H^6(S^6; \mathbf{Z})$ respectively, and define $\bar{x} := q^*(x)$ and $\bar{a} := q^*(a)$. For obvious dimensional reasons, the Chern classes $c_1(\bar{x})$ and $c_2(\bar{x})$ vanish. Moreover, one has $c_3(\bar{x}) = q^*(c_3(x)) = q^*(2a) = 0$ (see Proposition 2.4), hence $\text{c-dim}(\bar{x}) = 0$. On the other hand, we have $\gamma^1(\bar{x}) = \bar{x} \neq 0$, so $\gamma\text{-dim}(\bar{x}) \geq 1$; more precisely, $\gamma^2(\bar{x})$ is $q^*(-S(3,2) \cdot x) = q^*(-3x) = \bar{x} \neq 0$ and $\gamma^3(\bar{x}) = q^*(2S(3,3) \cdot x) = 0$, so $\gamma\text{-dim}(\bar{x}) = 2$. Consequently, M is a connected finite CW-complex with a strict inclusion

$$K_\gamma(M) \subsetneq K_c(M).$$

iii) Let $Z = Y \vee M$ be the wedge of the preceding two examples. It is a 7-dimensional finite connected CW-complex for which none of $K_\gamma(Z)$ and $K_c(Z)$ contains the other one. (The product $Y \times M$ would also do.)

To end the present section, we prove that the cones are semigroups and homotopy invariants.

PROPOSITION 3.5. *The positive cone, the γ-cone and the c-cone of a connected finite CW-complex X are sub-semigroups of $K(X)$ and homotopy invariants of X. Moreover, the positive cone is a sub-λ-semiring of $K(X)$.*

Proof. The homotopy invariance is obvious for the three cones. We have already mentioned in the preliminaries that the positive cone is a sub-semiring of $K(X)$. It is also clear that it is a sub-λ-semiring. The "exponentiality" of γ_t and of c (the total Chern class) immediately show that the γ-cone and the c-cone are sub-semigroups of $K(X)$. □

We do not know if in general the γ-cone and the c-cone are sub-λ-semirings of $K(X)$.

4. THE POSITIVE CONE OF THE SPHERES

We now intend to compute the positive cone of the spheres. For odd-dimensional spheres, there is nothing to do since $\widetilde{K}(S^{2n+1}) = 0$. Whereas for even-dimensional spheres, one has $\widetilde{K}(S^{2n}) = \mathbf{Z} \cdot x \cong \mathbf{Z}$, so we only have to compute $\text{g-dim}(lx)$ for all integers l.

By Proposition 2.4, we have

$$c(lx) = c(x)^l = (1 + (-1)^{n-1}(n-1)! \cdot a)^l = 1 + (-1)^{n-1} l(n-1)! \cdot a,$$