

## 6.5 MULTITEMPORAL WAVES

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **47 (2001)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **20.09.2024**

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern. Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden. Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

### **Haftungsausschluss**

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

scalar product  $(T, V) = \operatorname{Re} \sum \bar{T}_i V_i$  on  $\mathfrak{p}$  we have  $(T, V\lambda) = (T\bar{\lambda}, V)$ ,  $\lambda \in \mathbf{F}$ , therefore  $\mathfrak{s}^\perp$  is a  $\mathbf{F}$ -subspace of  $\mathfrak{p}$ .

An element  $k$  of  $K \cap H$  is characterized by  $k \in K$  and  $k \cdot \operatorname{Exp} \mathfrak{s} = \operatorname{Exp} \mathfrak{s}$ , i.e.  $k \cdot \mathfrak{s} = \mathfrak{s}$  (adjoint action). Let  $n', d'$  be the respective dimensions of  $\mathfrak{p}$  and  $\mathfrak{s}$  as  $\mathbf{F}$ -vector spaces. Taking a  $\mathbf{F}$ -basis of  $\mathfrak{p}$  according to the decomposition  $\mathfrak{p} = \mathfrak{s} \oplus \mathfrak{s}^\perp$ , it follows that

$$K = U(1; \mathbf{F}) \times U(n'; \mathbf{F}), \quad K \cap H = U(1; \mathbf{F}) \times U(d'; \mathbf{F}) \times U(n' - d'; \mathbf{F}).$$

But  $U(n' - d'; \mathbf{F})$  acts transitively on the unit sphere of  $\mathbf{F}^{n' - d'}$ , which implies our claim.

If  $T, T' \in \mathfrak{s}^\perp$  are two unit vectors, there exists  $k_o \in K \cap H$  such that  $k_o \cdot T = T'$ . Thus

$$\begin{aligned} R_{\exp tT'}^* v(gK) &= \int_K v(gkk_o \exp(tT)k_o^{-1}H) dk \\ &= \int_K v(gk \exp(tT)H) dk = R_{\exp tT}^* v(gK). \end{aligned}$$

In particular  $R_{\exp tT}^* v$  is an even function of  $t$ .

Going back to (23), we now take as  $(X_j)$  an orthonormal  $\mathbf{R}$ -basis of  $\mathfrak{p}$  according to the decomposition  $\mathfrak{p} = \mathfrak{s} \oplus \mathfrak{s}^\perp$ . The  $n - d$  basis vectors in  $\mathfrak{s}^\perp$  give the same contribution to the right hand side, whereas the  $d$  vectors in  $\mathfrak{s}$  generate one parameters subgroups of  $H$  and give no contribution; indeed  $\exp tV \cdot \operatorname{Exp} \mathfrak{s} = \operatorname{Exp} \mathfrak{s}$  for  $V \in \mathfrak{s}$ , since  $\mathfrak{s}$  is a Lie triple system by Section 4.3 c. This completes the proof.  $\square$

## 6.5 MULTITEMPORAL WAVES

We shall now deal with general invariant differential operators. As before  $G$  is a Lie group,  $H$  a closed subgroup,  $K$  a compact subgroup, and  $X = G/K$ ,  $Y = G/H$ . Let  $\mathfrak{g}, \mathfrak{h}, \mathfrak{k}$  be the respective Lie algebras, and  $\mathfrak{t}$  a vector subspace of  $\mathfrak{g}$  such that

$$\mathfrak{g} = (\mathfrak{k} + \mathfrak{h}) \oplus \mathfrak{t}.$$

Let  $K_1, \dots, K_p$  be a basis of  $\mathfrak{k}$ , complemented by  $H_1, \dots, H_q \in \mathfrak{h}$  so that the  $K_i$ 's and  $H_j$ 's are a basis of  $\mathfrak{k} + \mathfrak{h}$ , and let  $T_1, \dots, T_r$  be a basis of  $\mathfrak{t}$ . We shall use the same notations for the corresponding left-invariant vector fields on  $G$ , e.g.

$$K_i f(g) = \partial_s f(g \exp sK_i)|_{s=0},$$

with  $f \in C^\infty(G)$ ,  $g \in G$ ,  $s \in \mathbf{R}$ . We denote by  $\mathbf{D}(G)$  the algebra of all left invariant differential operators on  $G$ , by  $\mathbf{D}(G)^K$  the subalgebra of right

$K$ -invariant operators and by  $\mathbf{D}(X)$  the algebra of  $G$ -invariant differential operators on  $X$ . For  $s = (s_1, \dots, s_r) \in \mathbf{R}^r$ , let

$$t(s) = \exp s_1 T_1 \cdots \exp s_r T_r.$$

We recall that, for  $g, t \in G$ ,

$$R_t^* v(gK) = \int_K v(gktH) dk.$$

**THEOREM 17.** *Let  $G$  be a Lie group,  $H, K$  Lie subgroups, with  $K$  compact and  $X = G/K$ ,  $Y = G/H$ .*

(i) *For any  $P \in \mathbf{D}(X)$  there exists  $Q(\partial)$ , a constant coefficients differential operator on  $\mathbf{R}^r$ , with  $\text{order}(Q) \leq \text{order}(P)$ , such that for any  $v \in C^\infty(Y)$ ,  $x \in X$ ,*

$$(25) \quad PR^* v(x) = Q(\partial_s) R_{t(s)}^* v(x) \Big|_{s=0}.$$

(ii) *Assume furthermore that  $\mathfrak{t}$  is a Lie subalgebra of  $\mathfrak{g}$  with  $[\mathfrak{t}, \mathfrak{h}] \subset \mathfrak{h}$ , and let  $T$  denote the connected Lie subgroup of  $G$  with Lie algebra  $\mathfrak{t}$ . Then for any  $P \in \mathbf{D}(X)$  there exists a right-invariant differential operator  $Q$  on  $T$ , with  $\text{order}(Q) \leq \text{order}(P)$ , such that*

$$(26) \quad P_{(x)} R_t^* v(x) = Q_{(t)} R_t^* v(x)$$

for  $v \in C^\infty(Y)$ ; here  $P_{(x)}$  acts on the variable  $x \in X$  and  $Q_{(t)}$  acts on  $t \in T$ .

Thus  $R_t^* v(x)$ , as a function of  $(x, t) \in X \times T$ , solves the generalized “multitemporal” wave equation (26) with time variable in a multidimensional space. Similarly (25) can be viewed as a wave equation in the variables  $(x, s) \in X \times \mathbf{R}^r$  at the time  $s = 0$ .

*Proof.* In order to work on  $G$  rather than on its homogeneous spaces, we define  $w(g) = v(gH)$  and, for  $g, t \in G$ ,

$$(27) \quad F(g, t) = (R_t^* v)(gK) = \int_K w(gkt) dk,$$

so that  $F(gk, k'th) = F(g, t)$  for any  $k, k' \in K$ ,  $h \in H$ , and

$$F(g, e) = (R^* v)(gK) = \int_K w(gk) dk.$$

Let  $P \in \mathbf{D}(X)$  be given. Since  $K$  is compact the coset space  $X = G/K$  is reductive and there exists  $D \in \mathbf{D}(G)^K$  such that ([9], p. 285)

$$(28) \quad (Pf)(gK) = D_{(g)}(f(gK))$$

for  $f \in C^\infty(X)$ ,  $g \in G$ .

To transfer derivatives from  $g$  to  $t$  we observe that, by the invariance of  $D$  under left translation by  $gk$  and right translation by  $k$ ,

$$D_{(g)}w(gkt) = D_{(x)}w(gkxt)|_{x=e},$$

where  $g, x, t$  are variables in  $G$ . Integrating over  $K$  it follows that

$$(29) \quad D_{(g)}F(g, t) = D_{(x)}F(g, xt)|_{x=e},$$

By the Poincaré-Birkhoff-Witt theorem, the differential operators

$$K_1^{\beta_1} \cdots K_p^{\beta_p} T_1^{\alpha_1} \cdots T_r^{\alpha_r} H_1^{\gamma_1} \cdots H_q^{\gamma_q}$$

(where all exponents are positive integers) are a basis of  $\mathbf{D}(G)$ . Setting apart the terms with  $\beta = \gamma = 0$ , we can thus write, for some  $E_i, F_j \in \mathbf{D}(G)$  and some constant coefficients  $a_\alpha$ ,

$$(30) \quad D = D' + \sum_{i=1}^p K_i E_i + \sum_{j=1}^q F_j H_j, \quad D' = \sum_{\alpha} a_{\alpha_1 \dots \alpha_r} T_1^{\alpha_1} \cdots T_r^{\alpha_r}.$$

If we replace  $D_{(x)}$  by (30) in (29), the second term  $(K_i E_i)_{(x)} F(g, xt)|_{x=e}$  vanishes because  $K_i \in \mathfrak{k}$  and  $F(g, kxt) = F(g, t)$ . In the third term the left invariant vector field  $H_j \in \mathfrak{h}$  acts by

$$(H_j)_{(x)} F(g, xt) = \partial_s F(g, x \exp(sH_j) t)|_{s=0},$$

and this vanishes too whenever  $t$  normalizes  $H$ , because  $F(g, xth) = F(g, xt)$ .

Since  $t = e$  in case (i), or  $t \in T$  with  $Ht = tH$  in case (ii), we finally obtain for both cases (in multi-index notation)

$$(31) \quad \begin{aligned} D_{(g)}F(g, t) &= D'_{(x)}F(g, xt)|_{x=e} \\ &= \sum_{\alpha} a_{\alpha} \partial_s^{\alpha} F(g, (\exp s_1 T_1 \cdots \exp s_r T_r) t)|_{s=0} \\ &= \left( \sum_{\alpha} a_{\alpha} \partial_s^{\alpha} \right) F(g, t(s)t)|_{s=0}. \end{aligned}$$

Let the operator  $Q$  be defined by

$$Qf(t) = \sum_{\alpha} a_{\alpha} \partial_s^{\alpha} f(t(s)t)|_{s=0},$$

a right invariant differential operator on the group  $T$  in case (ii). The theorem now follows from (27), (28) and (31) in both cases (i) and (ii).  $\square$