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We briefly recall some classical semisimple notations, as used for instance in Helgason's books. Let  $\mathfrak{a}$  be a maximal abelian subspace of  $\mathfrak{p}$ . Related to the restricted root system of the pair  $(\mathfrak{g}, \mathfrak{a})$  are the eigenspaces  $\mathfrak{g}_\alpha$ , the *Iwasawa decomposition*  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$  of the Lie algebra and  $G = KAN$  for the group (unique decomposition of each element of  $G$  into a product of factors in the respective subgroups); the subgroups  $A$ , resp.  $N$ , of  $G$  are abelian, resp. nilpotent. The half sum of positive roots (counted with multiplicities) is a linear form  $\rho$  on  $\mathfrak{a}$ ; we write  $a^\rho = e^{\rho(\log a)}$  for  $a \in A$ . Let  $M$ , resp.  $M'$ , denote the centralizer, resp. normalizer, of  $A$  in  $K$ . Then  $W = M'/M$  is a finite group called the Weyl group.

Let  $y_o$  denote the orbit  $N \cdot x_o \subset X$ . The *horocycles* of  $X$  are the submanifolds  $g \cdot y_o$ , for  $g \in G$ . Since  $g \cdot y_o = y_o$  (globally) if and only if  $g \in MN$ , the space of all horocycles is  $Y = G/MN$ .

**e. ISOTROPIC RIEMANNIAN SYMMETRIC SPACES.** A Riemannian manifold  $X$  is called *isotropic* if, for every  $x \in X$  and every pair of unit tangent vectors  $V, W$  to  $X$  at  $x$ , there exists an isometry of  $X$  leaving  $x$  fixed and mapping  $V$  to  $W$ . The connected isotropic Riemannian manifolds are the Euclidean spaces  $\mathbf{R}^n$ , the *hyperbolic spaces* i.e. the Riemannian symmetric spaces of the noncompact type and of rank one ( $\dim \mathfrak{a} = 1$ ), and their compact analogues, spheres and projective spaces. The compact spaces will not be considered in this paper, so that most of our examples will be taken from the list

$$\mathbf{R}^n, H^n(\mathbf{R}), H^n(\mathbf{C}), H^n(\mathbf{H}), H^{16}(\mathbf{O}).$$

Among them we shall often restrict ourselves to the *classical hyperbolic spaces*  $H^n(\mathbf{F})$ , with  $\mathbf{F} = \mathbf{R}, \mathbf{C}$  or  $\mathbf{H}$ .

## 2. GEOMETRIC SETTING

### 2.1 DOUBLE FIBRATIONS OF HOMOGENEOUS SPACES

The general group-theoretic setting for Radon transforms, introduced by Helgason in the sixties, is motivated by the well-known example of points and hyperplanes in the Euclidean space  $\mathbf{R}^n$ . The set of points and the set of hyperplanes are both homogeneous spaces of the isometry group of  $\mathbf{R}^n$ , and it turns out that the fundamental "incidence" relation (a point  $x$  belongs to a hyperplane  $y$ ), as well as the defining integral of the Radon transform, have simple expressions in terms of Lie groups and invariant measures. This observation suggests considering the following general situation.

Let  $X$  and  $Y$  be two manifolds, with given origins  $x_o \in X$  and  $y_o \in Y$ , and assume a real Lie group  $G$  acts transitively on both manifolds  $X$  and  $Y$ . Two elements  $x \in X$  and  $y \in Y$  are said to be *incident* if there exists some  $g \in G$  such that  $x = g \cdot x_o$  and  $y = g \cdot y_o$ . Roughly speaking, if we think of  $g$  as a motion, this means that  $x$  and  $y$  have the same relative position as the origins  $x_o$  and  $y_o$ .

A more convenient formulation is obtained in terms of the isotropy subgroups  $K$ , resp.  $H$ , of  $x_o$ , resp.  $y_o$ , in  $G$ . They are closed Lie subgroups of  $G$ , and the manifolds  $X$ ,  $Y$  can be identified with the homogeneous spaces of left cosets  $G/K$ ,  $G/H$  respectively; in particular we may write  $x_o = K$ ,  $y_o = H$ ,  $g \cdot x_o = gK$ , etc. The points  $x = g'K \in X$  and  $y = g''H \in Y$  are then incident if and only if there exists  $g \in G$  such that  $g'K = g \cdot x_o = gK$  and  $g''H = g \cdot y_o = gH$ , in other words if the left cosets  $g'K$  and  $g''H$ , as subsets of  $G$ , are not disjoint (they meet at  $g$ ).

Given  $y = g''H$ , we see that  $x$  is incident to  $y$  if and only if  $x = g'hK$  for some  $h \in H$ . Given  $x = g'K$ , the point  $y$  is incident to  $x$  if and only if  $y = g'kH$  for some  $k \in K$ .

In the above example  $X$ , resp.  $Y$ , is the set of points, resp. hyperplanes, of  $\mathbf{R}^n$  and  $G$  is the group of all isometries. But hyperplanes can also be viewed as subsets of  $X = \mathbf{R}^n$ , and the incidence relation boils down to the familiar "the point  $x$  belongs to the hyperplane  $y$ " if and only if the chosen origin  $x_o$  belongs to the chosen origin  $y_o$ . Lemma 1 below extends this fact to Riemannian manifolds. More general incidence relations can be considered, however, and will be helpful in Section 6.

Clearly, the group  $G$  acts transitively on the subset  $Z$  of  $X \times Y$  consisting of all incident couples  $(x, y) = (g \cdot x_o, g \cdot y_o)$ , with  $K \cap H$  as the isotropy subgroup of the origin  $(x_o, y_o) \in Z$ . Thus  $Z = G/(K \cap H)$  can be endowed with a structure of manifold, and the present setting can be summarized by the following *double fibration of homogeneous spaces*

$$\begin{array}{ccc} Z = G/(K \cap H) & \subset & X \times Y \\ \downarrow & \searrow & \\ X = G/K & & Y = G/H, \end{array}$$

where the arrows denote the natural projections.

Radon transforms can be studied with more general double fibrations of manifolds  $X$ ,  $Y$ ,  $Z$  (without groups), as introduced by Gel'fand et al. [4]. We refer to Guillemin and Sternberg ([6], p. 340, 370) for their basic properties; this theory has been developed in several papers by Boman, Quinto, and others.

## 2.2 GROUP-THEORETIC RADON TRANSFORMS

Let  $G$  be a real Lie group and  $K$  a (closed) Lie subgroup, equipped with left-invariant Haar measures  $dg$ ,  $dk$  respectively. If the homogeneous space  $G/K$  admits a  $G$ -invariant measure  $d(gK)$ , the measures can then be normalized so that

$$\int_G f(g) dg = \int_{G/K} d(gK) \int_K f(gk) dk,$$

for any  $f \in C_c(G)$ . This applies in particular if  $K$  is compact (on invariant measures, see [9], Chap. I, § 1).

Throughout the paper  $G$  will be a Lie group,  $K$  a compact subgroup, and  $H$  a (closed) Lie subgroup of  $G$ . The Haar measure  $dk$  of  $K$  will be normalized by  $\int_K dk = 1$ .

Let  $u$  be a (complex-valued) function on  $X = G/K$ . Its Radon transform is the function  $Ru$  on  $Y = G/H$  defined by

$$Ru(gH) = \int_H u(ghK) dh,$$

for  $g \in G$ , whenever this makes sense (e.g. if  $u \in C_c(X)$ ). The left invariance of  $dh$  implies that the integral only depends on the left coset  $gH$  of  $g$ . Given  $y = gH$  in  $Y = G/H$ , the value  $Ru(y)$  is an integral of  $u$  over all  $x$  incident to  $y$ . A more precise statement can be given in the following important example.

EXAMPLE. Let  $X$  be a connected Riemannian manifold,  $G$  a transitive Lie group of isometries of  $X$  and  $K$  the isotropy subgroup of some origin  $x_o \in X$ ; then  $K$  is compact ([8], p. 204) and  $X = G/K$ . Let  $y_o$  be a given closed submanifold of  $X$ , containing  $x_o$ , and let  $Y$  be the set of all submanifolds  $y = g \cdot y_o$  of  $X$ , with  $g \in G$ .

The set  $H$  of all  $h \in G$  such that  $h \cdot y_o = y_o$  (i.e. the submanifold  $y_o$  is globally invariant under  $h$ ) is a closed Lie subgroup of  $G$ . Indeed if  $h_n \in H$  converges to  $h$  in  $G$ , for any  $x \in y_o$  the point  $\lim h_n \cdot x = h \cdot x$  belongs to  $y_o$ ; similarly  $h^{-1} \cdot x \in y_o$ , so that  $h \cdot y_o = y_o$ . Thus  $Y = G/H$  can be endowed with a structure of manifold and we obtain a double fibration of homogeneous spaces.

The following lemma allows one to compute the Radon transform without knowing  $H$  explicitly.

LEMMA 1. *Keeping the notation of this example, assume furthermore that  $y_o = G' \cdot x_o$  is a closed orbit of the origin  $x_o = K$  under some Lie subgroup  $G'$  of  $G$ .*

*Then  $G' \subset H \subset G'K$  and  $y_o = H \cdot x_o$ . The incidence relation between  $X = G/K$  and  $Y = G/H$  is simply  $x \in y$ . Besides, the left-invariant Haar measures  $dh, dg'$  of the groups  $H, G'$  can be normalized so that*

$$\begin{aligned} Ru(y) &= \int_H u(gh \cdot x_o) dh = \int_{G'} u(gg' \cdot x_o) dg' \\ &= \int_y u(x) dm_y(x), \end{aligned}$$

*where  $dm_y$  is the Riemannian measure induced by  $X$  on its submanifold  $y = g \cdot y_o$ .*

REMARK. The subgroup  $H$  can of course be strictly bigger than  $G'$ . This occurs for instance if  $y_o$  is a line in  $X = \mathbf{R}^n$  and  $G'$  is the group of translations along this line, or a horocycle in a Riemannian symmetric space  $X$  of the noncompact type (for which  $G' = N$  and  $H = MN = NM$  in the usual semisimple notations).

*Proof of Lemma 1.* If  $y_o = G' \cdot x_o$ , then  $H$  obviously contains  $G'$  and it follows that

$$y_o = G' \cdot x_o \subset H \cdot x_o \subset y_o,$$

whence  $H \cdot x_o = G' \cdot x_o$  and  $H \subset G'K$ .

A point  $x \in X$  is incident to  $y = g \cdot y_o \in Y$  if and only if there exists  $h \in H$  such that  $x = gh \cdot x_o$ , i.e.  $x \in gH \cdot x_o = g \cdot y_o = y$ .

An isometry  $g$  transforms the Riemannian measure of  $y_o$  into the Riemannian measure of  $y = g \cdot y_o$ , and it suffices to prove the integral formula for  $g = e$ . Now  $y_o = H \cdot x_o$  can be identified to the homogeneous space  $H/(H \cap K)$ , and  $dm_{y_o}$  (which is invariant under all isometries of  $y_o$ ) to an  $H$ -invariant measure on this space. The Haar measure  $dh$  can therefore be normalized so that the corresponding measure on  $H/(H \cap K)$  satisfies

$$\begin{aligned} \int_{y_o} u(x) dm_{y_o}(x) &= \int_{H/(H \cap K)} u(h \cdot x_o) d(h(H \cap K)) \\ &= \int_H u(h \cdot x_o) dh = Ru(y_o). \end{aligned}$$

The proof is similar for  $\int_{G'}$ , whence the lemma.  $\square$

Going back to general double fibrations, the *Radon dual transform* of a (continuous, say) function  $v$  on  $Y = G/H$  is the function on  $X = G/K$  defined by

$$R^*v(gK) = \int_K v(gkH) dk,$$

for  $g \in G$ , an integral of  $v$  over all  $y$  incident to  $x = gK$ . The word “dual” is of course motivated by the classical projective duality between points and hyperplanes in the basic example, but it also stems from the following proposition.

**PROPOSITION 2.** *Let  $X = G/K$  with  $K$  compact, and assume that  $Y = G/H$  has a  $G$ -invariant measure. Let  $u \in C_c(X)$ ,  $v \in C(Y)$ . Then  $Ru \in C_c(Y)$ ,  $R^*v \in C(X)$  and*

$$\int_X u(x) R^*v(x) dx = \int_Y Ru(y) v(y) dy = \int_Z u(x) v(y) dz,$$

where  $dx, dy, dz$  are the respective  $G$ -invariant measures on  $X, Y$  and  $Z = G/(K \cap H)$ .

In the latter integral  $u(x)v(y)$  is considered as a function of  $z = (x, y)$  on  $Z$  (Section 2.1). We omit the proof, a classical exercise on invariant integrals (cf. [9], p. 144 and [11], p. 41); all groups are assumed unimodular there, but the proof only uses the invariant measures on the homogeneous spaces, thus extends to the present situation.

Proposition 2 allows a natural extension of the transforms  $R$  and  $R^*$  to distributions. Given  $u \in \mathcal{E}'(X)$ , the distribution  $Ru \in \mathcal{E}'(Y)$  is defined by

$$\langle Ru, v \rangle = \langle u, R^*v \rangle,$$

for all test functions  $v \in C^\infty(Y)$ . Similarly, given  $v \in \mathcal{D}'(Y)$ , the distribution  $R^*v \in \mathcal{D}'(X)$  is defined by

$$\langle R^*v, u \rangle = \langle v, Ru \rangle,$$

for all  $u \in \mathcal{D}(X)$ . Again we refer to Helgason ([11], p. 42) for details, based on the compactness of  $K$ . These definitions do extend the Radon integrals for functions, as Proposition 2 shows, when identifying a function  $u$  with the distribution  $u(x) dx$ , and similarly for  $v$ .