2.2 DISCRETE THEOREM ON 6 AFFINE VERTICES

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THEOREM 2.2. *Every plane convex polygon P has at least 4 extremal triples of vertices.*

EXAMPLE 2.3. If P is a quadrilateral then the theorem holds tautologically since the $(i-1)^{\text{st}}$ vertex coincides with the $(i+3)^{\text{rd}}$ for every i.

REMARK 2.4. An alternative approach to discretization of the 4-vertex theorem consists in inscribing circles in consecutive triples of sides of a polygon (the centre of such a circle is the intersection point of the bisectors of consecutive angles of the polygon). Then a triple of sides $(\ell_i, \ell_{i+1}, \ell_{i+2})$ is said to be *extremal* if the lines ℓ_{i-1}, ℓ_{i+3} either both intersect the corresponding circle or both fail to intersect it. With this definition an analogue of Theorem 2.2 holds true [19, 16], and this, in the limit, also provides the smooth 4-vertex theorem.

Both formulations, concerning circumscribed or inscribed circles, make sense on the sphere. Moreover, they are equivalent via projective duality.

2.2 DISCRETE THEOREM ON 6 AFFINE VERTICES

Five generic points in the plane determine a conic. Considering the plane as an affine part of the projective plane, the complement of the conic has two connected components. Let P be a plane convex n-gon; throughout this section we assume that $n \geq 6$. As in the previous section, we introduce the following definition.

DEFINITION 2.5. Five consecutive vertices $V_i, \dots, V_{i+4}$ are said to be *extremal* if V_{i-1} and V_{i+5} lie on the same side of the conic through these 5 points (this does not exclude the case where V_{i-1} or V_{i+5} belongs to the conic).

If P is replaced by a smooth convex curve, and $V_i, \dots, V_{i+4}$ are infinitely close points, we recover the definition of an affine vertex. Hence the following theorem is a discrete version of the smooth theorem on 6 affine vertices.

THEOREM 2.6. *Every plane convex polygon P has at least 6 extremal quintuples of vertices.*

EXAMPLE 2.7. If P is a hexagon then the theorem holds tautologically for the same reason as in Example 2.3.

REMARK 2.8. On interchanging sides and vertices, and replacing circumscribed conics by inscribed ones, we arrive at a "dual" theorem. The latter is equivalent to Theorem 2.6 via projective duality – cf. Remark 2.4.

2.3 DISCRETE GHYS THEOREM

A discrete object of study in this section is a pair of cyclically ordered n-tuples $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$ in $\mathbf{RP}^1$ with $n \geq 4$. We choose an orientation of $\mathbf{RP}^1$ and assume that the cyclic ordering of each of the two n-tuples is induced by this orientation.

Recall that an ordered quadruple of distinct points in $\mathbf{RP}^1$ determines a number, the *cross-ratio*, which is a projective invariant. Choosing an affine parameter such that the points are given by real numbers $a < b < c < d$, the cross-ratio is

$$[a,b,c,d] = \frac{(c-a)(d-b)}{(b-a)(d-c)} . \tag{2.1}$$

DEFINITION 2.9. A triple of consecutive indices $(i, i+1, i+2)$ is said to be *extremal* if the difference of cross-ratios

$$[y_j, y_{j+1}, y_{j+2}, y_{j+3}] - [x_j, x_{j+1}, x_{j+2}, x_{j+3}] \tag{2.2}$$

changes sign as j varies from $i-1$ to i (this does not exclude the case where either of the differences vanishes).

THEOREM 2.10. *For every pair X, Y of n-tuples of points as above, there exist at least four extremal triples.*

EXAMPLE 2.11. If $n = 4$ then the theorem holds for a very simple reason. A cyclic permutation of four points induces the following transformation of their cross-ratio:

$$[x_4, x_1, x_2, x_3] = \frac{[x_1, x_2, x_3, x_4]}{[x_1, x_2, x_3, x_4] - 1} , \tag{2.3}$$

and this is an involution. Furthermore, if $a > b > 1$ then $a/(a-1) < b/(b-1)$. Therefore, each triple of indices is extremal.

Let us interpret Theorem 2.10 in geometrical terms like Theorems 2.2 and 2.6. There exists a unique projective transformation that carries x_i, x_{i+1}, x_{i+2} into y_i, y_{i+1}, y_{i+2}, respectively. The graph G of this transformation can be seen as a curve in $\mathbf{RP}^1 \times \mathbf{RP}^1$; the three points (x_i, y_i), (x_{i+1}, y_{i+1}), (x_{i+2}, y_{i+2}) lie