3. The top-dimensional obstruction

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **46 (2000)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **21.09.2024**

Theorem 4 in the general case is also due to the first author. The present joint note grew from the second author's observation that the original proof can be simplified and be made more geometric by appealing to considerations in [5].

This work was completed during a visit by the second author to the Forschungsinstitut für Mathematik (FIM) of the ETH Zürich. He thanks the FIM for its hospitality and support.

2. Stable almost complex structures

To derive Theorem 4(a) from Theorem 3(a) we merely have to show that the condition $w_8(M) \in \mathrm{Sq}^2 H^6(M;\mathbf{Z})$ is void in case I and equivalent to $\chi(M) \equiv 0 \mod 2$ (i.e. $w_8(M) = 0$) in case II.

Indeed, in case I we find an integral lift u of $w_2(M)$ whose free part is indivisible. Then there is a dual element $u' \in H^6(M;\mathbf{Z})$ such that $uu' = 1 \in H^8(M;\mathbf{Z})$. By the Wu formula it follows that

$$\mathrm{Sq}^2 H^6(M;\mathbf{Z}) = w_2(M)\, H^6(M;\mathbf{Z}) = \rho_2(u\, H^6(M;\mathbf{Z})) = H^8(M;\mathbf{Z}_2)\,.$$

In case II, on the other hand, we can lift $w_2(M)$ to a torsion class $u \in H^2(M;\mathbf{Z})$, thus

$$\mathrm{Sq}^2 H^6(M;\mathbf{Z}) = \rho_2(u\, H^6(M;\mathbf{Z})) = 0\,.$$

3. The top-dimensional obstruction

Assume we have an almost complex structure J_0 over M with a disc D^8 removed (which is homotopy equivalent to the 7-skeleton of M). Thinking of an almost complex structure J as a section of the SO_8/U_4-bundle associated to TM, we may interpret J_0 as such a section defined only over the 7-skeleton of M. The obstruction $\mathfrak{o}(M, J_0)$ to extending J_0 to an almost complex structure on M then lives in

$$H^8(M;\pi_7(\mathrm{SO}_8/U_4)) \cong \pi_7(\mathrm{SO}_8/U_4) \cong \mathbf{Z} \oplus \mathbf{Z}_2\,.$$

See [13] for references to the computations of the homotopy group above and others used below. The homotopy group involved here is in fact the first non-stable homotopy group of $\mathrm{SO}_8/\mathrm{U}_4$. As a consequence of this fact that the coefficient groups $\pi_i(\mathrm{SO}_8/\mathrm{U}_4)$ for the lower-dimensional obstructions are stable (that is, $\pi_i(\mathrm{SO}_8/\mathrm{U}_4) \cong \pi_i(\mathrm{SO}/\mathrm{U})$ for $i \leq 6$), any stable almost complex structure $\widetilde{J}$ on M induces a J_0 as described. The stabilizing map

$$
\begin{aligned}
S\colon \pi_7(\mathrm{SO}_8/\mathrm{U}_4) &\longrightarrow \pi_7(\mathrm{SO}/\mathrm{U}) \\
\mathbf{Z}\oplus\mathbf{Z}_2 &\longrightarrow \mathbf{Z}_2
\end{aligned}
$$

is surjective, cf. [5, p. 1213]. Define the splitting $\pi_7(\mathrm{SO}_8/U_4) \cong \mathbf{Z}\oplus\mathbf{Z}_2$ by identifying $\ker S$ with $\mathbf{Z}$. We can then write unambiguously

$$\mathfrak{o}(M,J_0) = \mathfrak{o}_0(M,J_0) + \mathfrak{o}_2(M,J_0) \in \mathbf{Z}\oplus\mathbf{Z}_2 .$$

Theorem II of [13] now states that

$$4\mathfrak{o}_0(M,J_0) = 2\chi(M) - 2c_1(J_0)\,c_3(J_0) + c_2(J_0)^2 - p_2(M) . \tag{1}$$

Given the relation between Pontrjagin and Chern classes, the obstruction $\mathfrak{o}_0(M,J_0)$, for J_0 induced from a $\widetilde{J}$ as above, can also be expressed as $\mathfrak{o}_0(M,J_0) = (\chi(M) - c_4(\widetilde{J}))/2$. So formula (1) can be regarded as a special case of the more general result in [17], already referred to in the introduction, that a stable a.c.s. $\widetilde{J}$ induces an a.c.s. if and only if the top-dimensional Chern class of $\widetilde{J}$ equals the Euler class of M.

We can also identify the stable part $\mathfrak{o}_2(M,J_0)$ of the obstruction.

LEMMA 9. *Given an almost complex structure J_0 over $M - D^8$, we have $c_1(J_0)\,c_3(J_0) \equiv 0 \mod 2$, and we can set*

$$\mathfrak{o}_2(M,J_0) = \rho_2\big(\chi(M) + \tfrac{1}{2}c_1(J_0)\,c_3(J_0)\big) . \tag{2}$$

Proof. The Wu relations $w_k = \sum_{i+j=k} \mathrm{Sq}^i(v_j)$ translate into $v_1 = w_1 = 0$ (M is orientable), $v_2 = w_2$, and $v_3 = w_3 = 0$ (since even $W_3 = \beta w_2$ is zero if there is an a.c.s. over the 3-skeleton). So we get further that $w_6 = \mathrm{Sq}^2 v_4$. Using the Adem relation $\mathrm{Sq}^2\mathrm{Sq}^2 = \mathrm{Sq}^3\mathrm{Sq}^1$ we obtain

$$
\begin{aligned}
\rho_2(c_1(J_0)\,c_3(J_0)) &= w_2 w_6 = \mathrm{Sq}^2 w_6 = \mathrm{Sq}^2\mathrm{Sq}^2 v_4 \\
&= \mathrm{Sq}^3\mathrm{Sq}^1 v_4 = v_3 \cup \mathrm{Sq}^1 v_4 = 0 .
\end{aligned}
$$

So it is possible to define $\mathfrak{o}_2(M,J_0)$ as stated, and by Theorem 3(a) this is indeed the obstruction to extending J_0 to a stable a.c.s. $\widetilde{J}$ over M. □

If $M = S^8$, then J_0 is unique up to homotopy (the 7-skeleton of S^8 is contractible), so we can write $\mathfrak{o}(S^8)$ for extending any J_0 to all of S^8. The formulae just stated give

$$\mathfrak{o}(S^8) = (1,0) \in \mathbf{Z} \oplus \mathbf{Z}_2 .$$

Results of Kahn [12] (which hold in a more general context, cf. the discussion and applications in [5]) now state the following: An almost complex structure J_0 on $M - D^8$ gives rise to a canonical almost complex structure $\overline{J}_0$ on $\overline{M} - D^8$, where $\overline{M}$ denotes M with reversed orientation, and we have

$$\mathfrak{o}(\overline{M}, \overline{J}_0) = -\mathfrak{o}(M, J_0) + \chi(M)\mathfrak{o}(S^8) .$$

Similarly, almost complex structures J_i on $M_i - D^8$, $i = 1,2$, give rise to a canonical almost complex structure $J_1 + J_2$ on the connected sum $M_1 \# M_2 - D^8$ such that

$$\mathfrak{o}(M_1 \# M_2, J_1 + J_2) = \mathfrak{o}(M_1, J_1) + \mathfrak{o}(M_2, J_2) - \mathfrak{o}(S^8) .$$

We now compute the obstruction $\mathfrak{o}$ for a few examples. First we consider the quaternionic projective plane $\mathbf{H}P^2$. By [8] the total Pontrjagin class of $\mathbf{H}P^2$ is

$$p(\mathbf{H}P^2) = (1+u)^6(1+4u)^{-1} = 1 + 2u + 7u^2 ,$$

where u is a suitable generator of $H^4(\mathbf{H}P^2; \mathbf{Z}) \cong \mathbf{Z}$. Since $\pi_3(\mathrm{SO}_8 / \mathrm{U}_4) \cong \pi_3(\mathrm{SO} / \mathrm{U}) = 0$, the structure group of $T\mathbf{H}P^2$ reduces to U_4 over the 4-skeleton $S^4 = \mathbf{H}P^1 \subset \mathbf{H}P^2$. Write J_0 for the resulting a.c.s. on $\mathbf{H}P^2 - D^8$. This structure is unique (up to homotopy), since reductions of the structure group over the 4-skeleton $M^{(4)}$ are classified by $H^4(M^{(4)}; \pi_4(\mathrm{SO}_8 / \mathrm{U}_4)) = 0$ (the coefficient group is trivial). The relation $p_1 = c_1^2 - 2c_2$ (which holds for any complex bundle) implies $c_2(J_0) = -u$. By (1) we find that

$$\begin{aligned} 4\,\mathfrak{o}_0(\mathbf{H}P^2, J_0) &= 2\chi(\mathbf{H}P^2) + c_2(J_0)^2 - p_2(\mathbf{H}P^2) \\ &= 6 - 6\langle u^2, [\mathbf{H}P^2] \rangle , \end{aligned}$$

where $[\mathbf{H}P^2]$ denotes the orientation generator of $H_8(\mathbf{H}P^2; \mathbf{Z})$. Now u^2 is a generator of $H^8(\mathbf{H}P^2; \mathbf{Z})$, so if we define the orientation of $\mathbf{H}P^2$ by the condition $\langle u^2, [\mathbf{H}P^2] \rangle = 1$, then $\mathfrak{o}_0(\mathbf{H}P^2, J_0) = 0$. With (2) we conclude

$$\mathfrak{o}(\mathbf{H}P^2, J_0) = (0,1) \in \mathbf{Z} \oplus \mathbf{Z}_2 .$$

Thus $\mathbf{H}P^2$ does not admit any a.c.s. (with either orientation, for with the opposite orientation we have even $\mathfrak{o}_0 \neq 0$).

We note in passing that the non-existence of an a.c.s. on $\mathbf{H}P^2$ is also a consequence of a more general result due to Hirzebruch [9, p. 124], which states that for an 8-dimensional almost complex manifold with $b_2 = 0$ the Euler characteristic has to be divisible by 6. This follows from a cobordism theoretic argument which shows that the condition $c_1c_3 + 2c_4 \equiv 0 \mod 12$, which holds for complex algebraic manifolds, must in fact hold for any almost complex 8-manifold.

Next we compute $\mathfrak{o}$ for $S^4 \times S^4$. Again we can find a unique a.c.s. J_0' over $S^4 \times S^4 - D^8$, since this retracts to the 4-skeleton $S^4 \vee S^4$. This manifold is stably parallelizable, so its total Pontrjagin class is equal to 1. It follows that $c_2(J_0') = 0$. Thus we find

$$\mathfrak{o}(S^4 \times S^4, J_0') = (2,0) \in \mathbf{Z} \oplus \mathbf{Z}_2 .$$

Again, we see that $S^4 \times S^4$ does not admit any a.c.s., independently of the orientation. This example shows that the condition $\chi \equiv \tau \mod 4$ is not sufficient for the existence of an a.c.s. in case II_0.

Proof of Proposition 6. We compute

$$\mathfrak{o}(\mathbf{H}P^2 \# \mathbf{H}P^2 \# S^4 \times S^4, J_0 + J_0 + J_0') = \\ (0,1) + (0,1) + (2,0) - 2 \cdot (1,0) = (0,0),$$

so $\mathbf{H}P^2 \# \mathbf{H}P^2 \# S^4 \times S^4$ admits an almost complex structure.

To prove the second part, we argue as follows. Eells-Kuiper [3] and Tamura [16] have constructed a family X_m of 3-connected 8-manifolds for integers m satisfying $m(m+1) \equiv 0 \mod 56$. Moreover, if $m \equiv 0 \mod 12$, then X_m is homotopy equivalent to $\mathbf{H}P^2$, and $X_0 = \mathbf{H}P^2$. So the X_m with $m = 12k$ and $k \equiv 0$ or $4 \mod 7$ constitute a family of 8-manifolds homotopy equivalent to $\mathbf{H}P^2$. They satisfy

$$p_1(X_m) = 2(2m+1)u ,$$

with u a generator of $H^4(X_m)$, and

$$p_2(X_m) = \tfrac{1}{7}(4(2m+1)^2 + 45)u^2 .$$

Hence, with J_m denoting the unique a.c.s. over the 4-skeleton $X_m^{(4)} \simeq S^4$, and with the orientation of X_m defined by u^2, a straightforward calculation yields

$$\mathfrak{o}_0(X_m, J_m) = 3m(m+1)/7 .$$

It follows that

$$\mathfrak{o}_0(X_m \# X_n \# S^4 \times S^4, J_m + J_n + J_0') = 3(m(m+1) + n(n+1))/7$$

(and $\mathfrak{o}_2 = 0$). With the given constraints on m and n, this can only be zero for $m = n = 0$ (even if we allow for the orientation of the summands to be changed).

For the allowed choices of m and n, this connected sum is homotopy equivalent to $\mathbf{H}P^2 \# \mathbf{H}P^2 \# S^4 \times S^4$. The fact that the homotopy equivalence $X_m, X_n \simeq X_0$ induces a homotopy equivalence of the connected sums is a simple consequence of the Whitehead theorem, since we are dealing with simply-connected manifolds. This concludes the proof of Proposition 6. □

4. Existence of almost complex structures

In this section we prove Theorem 4(b). We already know that condition (b)(i) is necessary. We now show that condition (b)(ii) is necessary.

Given an almost complex structure J on M, we have

$$2\chi(M) - 2c_1(J)\,c_3(J) + c_2(J)^2 - p_2(M) = 0$$

by (1). In the sequel we suppress M and J. In case II_0, c_1 is a torsion class, so this simplifies to

$$2\chi + c_2^2 - p_2 = 0\,.$$

Squaring the relation $p_1 = c_1^2 - 2c_2$, and again observing that c_1 is a torsion class, we get $p_1^2 = 4c_2^2$. Multiplying the equation above by 4 and substituting p_1^2 for $4c_2^2$ yields condition (b)(ii).

In fact, this argument also shows that (b)(ii) is a sufficient condition. By (a) we have a stable a.c.s. $\widetilde{J}$ on M and thus a corresponding J_0 as in Section 3. If condition (b)(ii) holds, then reversing the argument just given we find

$$4\mathfrak{o}_0(M, J_0) = 2\chi(M) + c_2(J_0)^2 - p_2(M) = 0\,.$$

Since J_0 is induced by $\widetilde{J}$, the stable part $\mathfrak{o}_2$ of the obstruction vanishes as well, so J_0 extends to an almost complex structure on M.

Next we prove that condition (b)(i) is sufficient for the existence of an a.c.s. We begin with a preparatory lemma.