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ALMOST COMPLEX STRUCTURES ON 8-MANIFOLDS

by Stefan MÜLLER and Hansjörg GEIGES

1. INTRODUCTION

Throughout this paper, M denotes a closed, connected, smooth and oriented manifold of even dimension $2n$. Our main goal is to give a criterion for the existence of almost complex structures on 8-dimensional manifolds.

Recall that an almost complex structure (a.c.s.) on M is an endomorphism J of the tangent bundle TM satisfying $J^2 = -1$. This gives TM the structure of a complex vector bundle, and we write $c_i(J)$ for its Chern classes. The orientation of M is required to coincide with the orientation induced by this complex vector bundle structure on TM.

Equivalently, we may think of an almost complex structure as a reduction of the structure group from the special orthogonal group SO_{2n} to the unitary group U_n. In enumeration questions one is of course interested in the classification of almost complex structures up to homotopy. As is well-known [15, §9.5], another equivalent way to think about an almost complex structure is as a cross-section of the $\mathrm{SO}_{2n}/\mathrm{U}_n$-bundle associated to TM. This viewpoint will be particularly relevant in Section 3.

Similarly, a stable almost complex structure on M is a reduction of the structure group of the stable tangent bundle of M from SO to U.

Thus, necessary conditions for the existence of a stable a.c.s. are the existence of integral lifts $c_i \in H^{2i}(M; \mathbf{Z})$ of the even Stiefel-Whitney classes $w_{2i}(M) \in H^{2i}(M; \mathbf{Z}_2)$, that is, $w_{2i}(M) = \rho_2 c_i$, with ρ_2 denoting mod 2 reduction. Given an a.c.s. J, the class $c_n(J)$ will be the Euler class of TM (which we may identify with the Euler characteristic $\chi(M)$). Indeed, a stable a.c.s. $\widetilde{J}$ induces an a.c.s. if and only if $c_n(\widetilde{J}) = \chi(M)$, see [17].

An oriented surface clearly carries a unique a.c.s. since $SO_2 = U_1$. For $\dim M = 4$, the existence of an integral lift of $w_2(M)$ is necessary and sufficient for the existence of a stable a.c.s., and this condition is always satisfied, see [11]. Furthermore, the integral lifts c_1 of $w_2(M)$ completely classify the stable a.c.s. With $\tau(M)$ denoting the signature of M, a well-known result of Wu (cf. [11]) asserts that a.c.s. on M are classified by those integral lifts c_1 that satisfy the signature formula

$$\langle c_1^2, [M] \rangle = 3\tau(M) + 2\chi(M).$$

A reformulation of this theorem was found independently by Dessai [2] and the first author (unpublished). Write b_i for the Betti numbers of M, and b_2^+ (resp. b_2^-) for the dimension of the positive (resp. negative) eigenspace of the intersection form Q on $H_2(M;\mathbf{Z})$. Then, observing that $\chi(M) + \tau(M) = 2(1 - b_1 + b_2^+)$, Dessai's Theorem 1.4 can be stated as follows.

THEOREM 1 (Dessai). *An oriented 4-manifold M admits an almost complex structure if and only if $\chi(M) + \tau(M) \equiv 0 \mod 4$ and one of the following conditions is satisfied:*

(i) *Q is indefinite.*

(ii) *Q is positive definite and $b_1 - b_2 \leq 1$.*

(iii) *Q is negative definite and, in case $b_2 \leq 2$, $4(b_1 - 1) + b_2$ is the sum of b_2 integer squares.*

REMARK. Observe that in case (iii) with b_2 equal to 1 or 2, the condition $\chi(M) + \tau(M) \equiv 0 \mod 4$ is implied by the other conditions stated.

The advantage of this formulation over that of Wu lies in the fact that the existence of an a.c.s. is expressed solely in terms of topological invariants of M rather than by requiring the existence of a solution to the signature formula in the potentially infinite set $\rho_2^{-1}(w_2(M)) \subset H^2(M;\mathbf{Z})$.

Note that for manifolds of dimension $4k$, the condition

$$\chi(M) \equiv (-1)^k \tau(M) \mod 4$$

is necessary for the existence of an a.c.s., as was observed by Hirzebruch [10, p. 777]. This follows from an integrality argument involving the Todd genus.

Dessai also gives a finiteness criterion for a.c.s. The following is a direct consequence of [2, Thm. 2.2].

PROPOSITION 2 (Dessai). *Let M be a 4-manifold admitting an a.c.s. There exist only finitely many a.c.s. on M if and only if one of the following conditions holds:*

(i) *The intersection form Q of M is definite.*

(ii) *Q is indefinite, $b_1 \neq 2$, $b_2 = 2$.*

Observe that the obstructions to the existence of an a.c.s. on M (with $\dim M = 2n$) lie in $H^k(M; \pi_{k-1}(\mathrm{SO}_{2n} / \mathrm{U}_n))$. For $\dim M = 6$, the only non-zero coefficient group here is $\pi_2(\mathrm{SO}_6 / U_3) \cong \mathbf{Z}$ (cf. [13]). The obstruction to the existence of an a.c.s. on a 6-manifold has been identified as the third integral Stiefel-Whitney class $W_3(M) = \beta w_2(M) \in H^3(M; \mathbf{Z})$, where β denotes the Bockstein homomorphism induced by the coefficient sequence $\mathbf{Z} \overset{2}{\rightarrowtail} \mathbf{Z} \twoheadrightarrow \mathbf{Z}_2$ (notice that $W_3(M) = 0$ is equivalent to the existence of an integral lift of $w_2(M)$). Indeed, $W_3(M)$ is the first obstruction to the existence of an a.c.s. in any dimension, see [13]. Furthermore, homotopy classes of a.c.s. on a 6-manifold are classified by the integral lifts c_1 of $w_2(M)$, cf. [4].

The corresponding existence result for 8-dimensional manifolds is due to Heaps [6]. Write Sq^2 for the Steenrod square and $p_i(M)$ for the Pontrjagin classes of M. In the sequel, cohomology classes in $H^8(M; \mathbf{Z})$ are usually interpreted as integers under the evaluation on the fundamental cycle $[M]$.

THEOREM 3 (Heaps). (a) *An oriented 8-dimensional manifold M admits a stable almost complex structure if and only if $\beta w_2(M) = 0$ and $w_8(M) \in \mathrm{Sq}^2 H^6(M; \mathbf{Z})$. In this case, any integer lift of $w_2(M)$ can be realized as $c_1(J)$ of some stable almost complex structure J. Furthermore, any pair $(u, v) \in H^2(M; \mathbf{Z}) \times H^6(M; \mathbf{Z})$ with $(\rho_2(u), \rho_2(v)) = (w_2(M), w_6(M))$ can be realized as $(c_1(J), c_3(J))$ for some stable almost complex structure J provided $2\chi(M) + uv \equiv 0 \mod 4$.*

(b) *M admits an almost complex structure if and only if the conditions in* (a) *are satisfied and if there is a pair (u, v) as in* (a) *with*

$$8\chi(M) - 4p_2(M) + p_1^2(M) = 8uv - u^4 + 2u^2 p_1(M) .$$

Implicit in statement (a) is a result of Massey that $\beta w_6(M) = 0$ for any 8-dimensional M. Heaps also proves a corresponding statement for 10-manifolds under some additional topological assumptions, see also [2].

Our following main result stands in the same relation to that of Heaps as Dessai's theorem to that of Wu. Write $TH^2(M)$ for the torsion subgroup of $H^2(M;\mathbf{Z})$. We distinguish three cases:

- case I: $\rho_2^{-1}(w_2(M)) \cap TH^2(M) = \varnothing$.
- case II_+: $\rho_2^{-1}(w_2(M)) \cap TH^2(M) \neq \varnothing$ and $b_2(M) > 0$.
- case II_0 : $\rho_2^{-1}(w_2(M)) \cap TH^2(M) \neq \varnothing$ and $b_2(M) = 0$.

Notice that case I implies that $w_2(M) \neq 0$. An example that the converse is false is provided by $M = E \times S^4$, with E denoting an Enriques surface. Here $w_2(M) \neq 0$, but we are in case II_+.

THEOREM 4. (a) *An oriented* 8*-dimensional manifold* M *admits a stable almost complex structure if and only if* $\beta w_2(M) = 0$ *and, in case* II, $\chi(M) \equiv 0 \mod 2$.

(b) M *admits an almost complex structure if and only if the conditions of* (a) *hold and*

(i) $\chi(M) \equiv \tau(M) \mod 4$ *in cases* I *and* II_+,

(ii) $8\chi(M) - 4p_2(M) + p_1^2(M) = 0$ *in case* II_0.

REMARK. By the observation of Hirzebruch mentioned above, the condition $\chi(M) \equiv \tau(M) \mod 4$ is certainly necessary for the existence of an a.c.s. on an 8-manifold. This condition is implied by (b)(ii).

Here is a simple example. Give $\mathbf{C}P^4$ its natural orientation induced by the complex structure, and write $\overline{\mathbf{C}P^4}$ for the same manifold with the opposite orientation. Consider the connected sum $M = \#_r\mathbf{C}P^4 \#_s\overline{\mathbf{C}P^4}$. Then we are in case I and $\chi(M) - \tau(M) = 2r + 4s + 2$. So M admits an a.c.s. if and only if r is odd. There is an analogous statement for $\#_r\mathbf{C}P^2 \#_s\overline{\mathbf{C}P^2}$, see [1].

Theorem 4 will be derived from some explicit calculations of obstruction classes. Only Theorem 3(a), but not 3(b), will be used for the proof of Theorem 4. The following is an immediate corollary.

COROLLARY 5. *On* 8*-dimensional manifolds* M *with* $b_2 > 0$ *the existence of an a.c.s. depends only on the oriented homotopy type of* M. $\square$

This is false if $b_2 = 0$.

PROPOSITION 6. *The manifold* $M = \mathbf{H}P^2 \# \mathbf{H}P^2 \# S^4 \times S^4$ *admits an almost complex structure. There are 8-manifolds homotopy equivalent to* M *which do not admit any almost complex structures.*

More generally, Kahn [12, Cor. 6] has shown, for every $k \geq 1$, the existence of pairs of closed, connected, oriented manifolds M_1, M_2 of dimension $8k$ such that M_1 and M_2 have the same oriented homotopy type, but only one of them admits an almost complex structure. According to the results already mentioned, dimension 8 is indeed the smallest dimension where this phenomenon can occur.

A further simple corollary concerns the compatibility with different choices of orientation.

COROLLARY 7. *In case* I, *if* M *admits an a.c.s., then* $\overline{M}$ *(i.e.* M *with reversed orientation) admits an a.c.s. if and only if* $\chi(M) \equiv 0 \mod 2$.

In case II_+, *either both* M *and* $\overline{M}$ *or none of them admits an a.c.s.*

In case II_0, *if* M *admits an a.c.s., then* $\overline{M}$ *admits an a.c.s. if and only if* $\chi(M) = 0$. $\square$

See [1] for related statements in the 4-dimensional situation.

Using the first author's original method of proof (cf. the acknowledgements below), it is possible to determine the set of all pairs (u, v) satisfying the conditions of Theorem 3; see [14] for details in the torsion-free case. These results lead to the following finiteness criterion, which is equivalent to the corresponding case in Theorems 2.2 and 2.3 of [2].

PROPOSITION 8. *Assume the oriented* 8*-manifold admits an almost complex structure. Then the number of a.c.s. on* M *is finite if and only if* $b_2 = 0$, *or* $b_1 = 1$ *and* $8\chi(M) - 4p_2(M) + p_1^2(M) \neq 0$. $\square$

We note in passing that applying this result to complete intersections corrects an error in [7]; see [14, §4.5.2.] and [2, §2.3].

ACKNOWLEDGEMENTS. Theorem 4 was proved in the torsion-free case in the first author's Ph.D. thesis [14] written under the guidance of Ch. Okonek. The first author thanks him for his support, and also P. Lupascu for many fruitful conversations. The work [14] was partially supported by the Schweizerischer Nationalfonds SNF under grant no. 2000-045209.95/1.

Theorem 4 in the general case is also due to the first author. The present joint note grew from the second author's observation that the original proof can be simplified and be made more geometric by appealing to considerations in [5].

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2. Stable almost complex structures

To derive Theorem 4(a) from Theorem 3(a) we merely have to show that the condition $w_8(M) \in \mathrm{Sq}^2 H^6(M;\mathbf{Z})$ is void in case I and equivalent to $\chi(M) \equiv 0 \mod 2$ (i.e. $w_8(M) = 0$) in case II.

Indeed, in case I we find an integral lift u of $w_2(M)$ whose free part is indivisible. Then there is a dual element $u' \in H^6(M;\mathbf{Z})$ such that $uu' = 1 \in H^8(M;\mathbf{Z})$. By the Wu formula it follows that

$$\mathrm{Sq}^2 H^6(M;\mathbf{Z}) = w_2(M)\, H^6(M;\mathbf{Z}) = \rho_2(u\, H^6(M;\mathbf{Z})) = H^8(M;\mathbf{Z}_2)\,.$$

In case II, on the other hand, we can lift $w_2(M)$ to a torsion class $u \in H^2(M;\mathbf{Z})$, thus

$$\mathrm{Sq}^2 H^6(M;\mathbf{Z}) = \rho_2(u\, H^6(M;\mathbf{Z})) = 0\,.$$

3. The top-dimensional obstruction

Assume we have an almost complex structure J_0 over M with a disc D^8 removed (which is homotopy equivalent to the 7-skeleton of M). Thinking of an almost complex structure J as a section of the SO_8/U_4-bundle associated to TM, we may interpret J_0 as such a section defined only over the 7-skeleton of M. The obstruction $\mathfrak{o}(M, J_0)$ to extending J_0 to an almost complex structure on M then lives in

$$H^8(M;\pi_7(\mathrm{SO}_8/U_4)) \cong \pi_7(\mathrm{SO}_8/U_4) \cong \mathbf{Z} \oplus \mathbf{Z}_2\,.$$

See [13] for references to the computations of the homotopy group above and others used below. The homotopy group involved here is in fact the first non-stable homotopy group of $\mathrm{SO}_8/\mathrm{U}_4$. As a consequence of this fact that the coefficient groups $\pi_i(\mathrm{SO}_8/\mathrm{U}_4)$ for the lower-dimensional obstructions are stable (that is, $\pi_i(\mathrm{SO}_8/\mathrm{U}_4) \cong \pi_i(\mathrm{SO}/\mathrm{U})$ for $i \leq 6$), any stable almost complex structure $\widetilde{J}$ on M induces a J_0 as described. The stabilizing map

$$\begin{aligned} S\colon \pi_7(\mathrm{SO}_8/\mathrm{U}_4) &\longrightarrow \pi_7(\mathrm{SO}/\mathrm{U}) \\ \mathbf{Z}\oplus\mathbf{Z}_2 &\longrightarrow \mathbf{Z}_2 \end{aligned}$$

is surjective, cf. [5, p. 1213]. Define the splitting $\pi_7(\mathrm{SO}_8/U_4) \cong \mathbf{Z}\oplus\mathbf{Z}_2$ by identifying $\ker S$ with $\mathbf{Z}$. We can then write unambiguously

$$\mathfrak{o}(M, J_0) = \mathfrak{o}_0(M, J_0) + \mathfrak{o}_2(M, J_0) \in \mathbf{Z}\oplus\mathbf{Z}_2 .$$

Theorem II of [13] now states that

$$4\mathfrak{o}_0(M, J_0) = 2\chi(M) - 2c_1(J_0)\,c_3(J_0) + c_2(J_0)^2 - p_2(M) . \tag{1}$$

Given the relation between Pontrjagin and Chern classes, the obstruction $\mathfrak{o}_0(M, J_0)$, for J_0 induced from a $\widetilde{J}$ as above, can also be expressed as $\mathfrak{o}_0(M, J_0) = (\chi(M) - c_4(\widetilde{J}))/2$. So formula (1) can be regarded as a special case of the more general result in [17], already referred to in the introduction, that a stable a.c.s. $\widetilde{J}$ induces an a.c.s. if and only if the top-dimensional Chern class of $\widetilde{J}$ equals the Euler class of M.

We can also identify the stable part $\mathfrak{o}_2(M, J_0)$ of the obstruction.

LEMMA 9. *Given an almost complex structure J_0 over $M - D^8$, we have $c_1(J_0)\,c_3(J_0) \equiv 0 \mod 2$, and we can set*

$$\mathfrak{o}_2(M, J_0) = \rho_2\big(\chi(M) + \tfrac{1}{2}c_1(J_0)\,c_3(J_0)\big) . \tag{2}$$

Proof. The Wu relations $w_k = \sum_{i+j=k} \mathrm{Sq}^i(v_j)$ translate into $v_1 = w_1 = 0$ (M is orientable), $v_2 = w_2$, and $v_3 = w_3 = 0$ (since even $W_3 = \beta w_2$ is zero if there is an a.c.s. over the 3-skeleton). So we get further that $w_6 = \mathrm{Sq}^2 v_4$. Using the Adem relation $\mathrm{Sq}^2\mathrm{Sq}^2 = \mathrm{Sq}^3\mathrm{Sq}^1$ we obtain

$$\begin{aligned} \rho_2(c_1(J_0)\,c_3(J_0)) &= w_2 w_6 = \mathrm{Sq}^2 w_6 = \mathrm{Sq}^2\mathrm{Sq}^2 v_4 \\ &= \mathrm{Sq}^3\mathrm{Sq}^1 v_4 = v_3 \cup \mathrm{Sq}^1 v_4 = 0 . \end{aligned}$$

So it is possible to define $\mathfrak{o}_2(M, J_0)$ as stated, and by Theorem 3(a) this is indeed the obstruction to extending J_0 to a stable a.c.s. $\widetilde{J}$ over M. □

If $M = S^8$, then J_0 is unique up to homotopy (the 7-skeleton of S^8 is contractible), so we can write $\mathfrak{o}(S^8)$ for extending any J_0 to all of S^8. The formulae just stated give

$$\mathfrak{o}(S^8) = (1,0) \in \mathbf{Z} \oplus \mathbf{Z}_2 .$$

Results of Kahn [12] (which hold in a more general context, cf. the discussion and applications in [5]) now state the following: An almost complex structure J_0 on $M - D^8$ gives rise to a canonical almost complex structure $\overline{J}_0$ on $\overline{M} - D^8$, where $\overline{M}$ denotes M with reversed orientation, and we have

$$\mathfrak{o}(\overline{M}, \overline{J}_0) = -\mathfrak{o}(M, J_0) + \chi(M)\mathfrak{o}(S^8) .$$

Similarly, almost complex structures J_i on $M_i - D^8$, $i = 1,2$, give rise to a canonical almost complex structure $J_1 + J_2$ on the connected sum $M_1 \# M_2 - D^8$ such that

$$\mathfrak{o}(M_1 \# M_2, J_1 + J_2) = \mathfrak{o}(M_1, J_1) + \mathfrak{o}(M_2, J_2) - \mathfrak{o}(S^8) .$$

We now compute the obstruction $\mathfrak{o}$ for a few examples. First we consider the quaternionic projective plane $\mathbf{H}P^2$. By [8] the total Pontrjagin class of $\mathbf{H}P^2$ is

$$p(\mathbf{H}P^2) = (1+u)^6(1+4u)^{-1} = 1 + 2u + 7u^2 ,$$

where u is a suitable generator of $H^4(\mathbf{H}P^2; \mathbf{Z}) \cong \mathbf{Z}$. Since $\pi_3(\mathrm{SO}_8 / \mathrm{U}_4) \cong \pi_3(\mathrm{SO} / \mathrm{U}) = 0$, the structure group of $T\mathbf{H}P^2$ reduces to U_4 over the 4-skeleton $S^4 = \mathbf{H}P^1 \subset \mathbf{H}P^2$. Write J_0 for the resulting a.c.s. on $\mathbf{H}P^2 - D^8$. This structure is unique (up to homotopy), since reductions of the structure group over the 4-skeleton $M^{(4)}$ are classified by $H^4(M^{(4)}; \pi_4(\mathrm{SO}_8 / \mathrm{U}_4)) = 0$ (the coefficient group is trivial). The relation $p_1 = c_1^2 - 2c_2$ (which holds for any complex bundle) implies $c_2(J_0) = -u$. By (1) we find that

$$\begin{aligned} 4\,\mathfrak{o}_0(\mathbf{H}P^2, J_0) &= 2\chi(\mathbf{H}P^2) + c_2(J_0)^2 - p_2(\mathbf{H}P^2) \\ &= 6 - 6\langle u^2, [\mathbf{H}P^2] \rangle , \end{aligned}$$

where $[\mathbf{H}P^2]$ denotes the orientation generator of $H_8(\mathbf{H}P^2; \mathbf{Z})$. Now u^2 is a generator of $H^8(\mathbf{H}P^2; \mathbf{Z})$, so if we define the orientation of $\mathbf{H}P^2$ by the condition $\langle u^2, [\mathbf{H}P^2] \rangle = 1$, then $\mathfrak{o}_0(\mathbf{H}P^2, J_0) = 0$. With (2) we conclude

$$\mathfrak{o}(\mathbf{H}P^2, J_0) = (0,1) \in \mathbf{Z} \oplus \mathbf{Z}_2 .$$

Thus $\mathbf{H}P^2$ does not admit any a.c.s. (with either orientation, for with the opposite orientation we have even $\mathfrak{o}_0 \neq 0$).

We note in passing that the non-existence of an a.c.s. on $\mathbf{H}P^2$ is also a consequence of a more general result due to Hirzebruch [9, p. 124], which states that for an 8-dimensional almost complex manifold with $b_2 = 0$ the Euler characteristic has to be divisible by 6. This follows from a cobordism theoretic argument which shows that the condition $c_1c_3 + 2c_4 \equiv 0 \mod 12$, which holds for complex algebraic manifolds, must in fact hold for any almost complex 8-manifold.

Next we compute $\mathfrak{o}$ for $S^4 \times S^4$. Again we can find a unique a.c.s. J_0' over $S^4 \times S^4 - D^8$, since this retracts to the 4-skeleton $S^4 \vee S^4$. This manifold is stably parallelizable, so its total Pontrjagin class is equal to 1. It follows that $c_2(J_0') = 0$. Thus we find

$$\mathfrak{o}(S^4 \times S^4, J_0') = (2, 0) \in \mathbf{Z} \oplus \mathbf{Z}_2 .$$

Again, we see that $S^4 \times S^4$ does not admit any a.c.s., independently of the orientation. This example shows that the condition $\chi \equiv \tau \mod 4$ is not sufficient for the existence of an a.c.s. in case II_0.

Proof of Proposition 6. We compute

$$\mathfrak{o}(\mathbf{H}P^2 \# \mathbf{H}P^2 \# S^4 \times S^4, J_0 + J_0 + J_0') =$$
$$(0,1) + (0,1) + (2,0) - 2 \cdot (1,0) = (0,0),$$

so $\mathbf{H}P^2 \# \mathbf{H}P^2 \# S^4 \times S^4$ admits an almost complex structure.

To prove the second part, we argue as follows. Eells-Kuiper [3] and Tamura [16] have constructed a family X_m of 3-connected 8-manifolds for integers m satisfying $m(m+1) \equiv 0 \mod 56$. Moreover, if $m \equiv 0 \mod 12$, then X_m is homotopy equivalent to $\mathbf{H}P^2$, and $X_0 = \mathbf{H}P^2$. So the X_m with $m = 12k$ and $k \equiv 0$ or $4 \mod 7$ constitute a family of 8-manifolds homotopy equivalent to $\mathbf{H}P^2$. They satisfy

$$p_1(X_m) = 2(2m+1)u ,$$

with u a generator of $H^4(X_m)$, and

$$p_2(X_m) = \tfrac{1}{7}(4(2m+1)^2 + 45)u^2 .$$

Hence, with J_m denoting the unique a.c.s. over the 4-skeleton $X_m^{(4)} \simeq S^4$, and with the orientation of X_m defined by u^2, a straightforward calculation yields

$$\mathfrak{o}_0(X_m, J_m) = 3m(m+1)/7 .$$

It follows that

$$\mathfrak{o}_0(X_m \# X_n \# S^4 \times S^4, J_m + J_n + J_0') = 3(m(m+1) + n(n+1))/7$$

(and $\mathfrak{o}_2 = 0$). With the given constraints on m and n, this can only be zero for $m = n = 0$ (even if we allow for the orientation of the summands to be changed).

For the allowed choices of m and n, this connected sum is homotopy equivalent to $\mathbf{H}P^2 \# \mathbf{H}P^2 \# S^4 \times S^4$. The fact that the homotopy equivalence $X_m, X_n \simeq X_0$ induces a homotopy equivalence of the connected sums is a simple consequence of the Whitehead theorem, since we are dealing with simply-connected manifolds. This concludes the proof of Proposition 6. $\square$

4. Existence of almost complex structures

In this section we prove Theorem 4(b). We already know that condition (b) (i) is necessary. We now show that condition (b) (ii) is necessary.

Given an almost complex structure J on M, we have

$$2\chi(M) - 2c_1(J)\, c_3(J) + c_2(J)^2 - p_2(M) = 0$$

by (1). In the sequel we suppress M and J. In case II_0, c_1 is a torsion class, so this simplifies to

$$2\chi + c_2^2 - p_2 = 0\,.$$

Squaring the relation $p_1 = c_1^2 - 2c_2$, and again observing that c_1 is a torsion class, we get $p_1^2 = 4c_2^2$. Multiplying the equation above by 4 and substituting p_1^2 for $4c_2^2$ yields condition (b) (ii).

In fact, this argument also shows that (b) (ii) is a sufficient condition. By (a) we have a stable a.c.s. $\widetilde{J}$ on M and thus a corresponding J_0 as in Section 3. If condition (b) (ii) holds, then reversing the argument just given we find

$$4\mathfrak{o}_0(M, J_0) = 2\chi(M) + c_2(J_0)^2 - p_2(M) = 0\,.$$

Since J_0 is induced by $\widetilde{J}$, the stable part $\mathfrak{o}_2$ of the obstruction vanishes as well, so J_0 extends to an almost complex structure on M.

Next we prove that condition (b) (i) is sufficient for the existence of an a.c.s. We begin with a preparatory lemma.

LEMMA 10. *Let M be an 8-manifold with $b_2 > 0$ which satisfies the assumptions of Theorem 4(a). Then there is a family $\widetilde{J}_k$, $k \in \mathbf{Z}$, of stable almost complex structures on M such that*

$$c_1(\widetilde{J}_k)\, c_3(\widetilde{J}_k) = c_1(\widetilde{J}_0)c_3(\widetilde{J}_0) + 4k$$

and

$$c_2(\widetilde{J}_k) = c_2(\widetilde{J}_0) \ \textit{modulo 2-torsion},$$

so in particular $c_2(\widetilde{J}_k)^2 = c_2(\widetilde{J}_0)^2$.

Proof. Case I: By Theorem 3(a) we can find a stable a.c.s. $\widetilde{J}_0$ such that the free part x of $c_1(\widetilde{J}_0)$ is indivisible. By Poincaré duality there is an element $x' \in H^6(M;\mathbf{Z})$ such that $c_1(\widetilde{J}_0)x' = xx' = 1$. Then the pair $(c_1(\widetilde{J}_0), c_3(\widetilde{J}_0) + 4kx')$, $k \in \mathbf{Z}$, still satisfies the assumptions of Theorem 3(a), so there are stable almost complex structures $\widetilde{J}_k$, $k \in \mathbf{Z}$, with

$$c_1(\widetilde{J}_k) = c_1(\widetilde{J}_0) \quad \text{and} \quad c_3(\widetilde{J}_k) = c_3(\widetilde{J}_0) + 4kx'.$$

This is the desired family, since the relation $p_1 = c_1^2 - 2c_2$ shows that c_2 is determined modulo 2-torsion by c_1 and p_1.

Case II$_+$: Let $\widetilde{J}_0$ be a stable a.c.s. such that $c_1(\widetilde{J}_0)$ is torsion. Let x be an indivisible element of $H^2(M;\mathbf{Z})$ and x' a dual element of $H^6(M;\mathbf{Z})$, i.e. $xx' = 1$ and $yx' = 0$ for all y in a complement of $\mathbf{Z}x \subset H^2(M;\mathbf{Z})$ (of course x' depends on the choice of this complement). Then by Theorem 3(a) there exists a stable a.c.s. $\widetilde{J}_k$ with

$$c_1(\widetilde{J}_k) = c_1(\widetilde{J}_0) + 2x \quad \text{and} \quad c_3(\widetilde{J}_k) = c_3(\widetilde{J}_0) + 2kx'.$$

This family has the desired properties. □

Now, assuming that condition (b)(i) is satisfied, we choose a stable a.c.s. $\widetilde{J}$ on M as in the proof of the preceding lemma. Hence we get an a.c.s. J_1 on $M - D^8$ with

$$\mathfrak{o}(M, J_1) = (a_1, 0) \in \mathbf{Z} \oplus \mathbf{Z}_2.$$

If a_1 is even, then by formulae (1) and (2) for $\mathfrak{o}$ and the preceding lemma, we can find a different a.c.s. J_1' on $M - D^8$ with $\mathfrak{o}(M, J_1') = (0,0)$, so that J_1' extends to an a.c.s. on M.

We complete the proof of Theorem 4 by showing that a_1 has to be even. Since

$$\mathfrak{o}(M \# \mathbf{H}P^2 \# \mathbf{H}P^2, J_1 + J_0 + J_0) = (a_1 - 2, 0),$$

we can find an 8-manifold M_2 and an a.c.s. J_2 on $M_2 - D^8$ with

$$\mathfrak{o}(M_2, J_2) = (-a_2, 0)$$

with $a_2 > 0$ of the same parity as a_1, and

$$\chi(M_2) - \tau(M_2) = \chi(M) - \tau(M) \equiv 0 \mod 4\,.$$

Then

$$\mathfrak{o}(M_2 \# a_2 S^4 \times S^4, J_2 + a_2 J_0') = (0, 0)\,.$$

So $M_2 \# a_2 S^4 \times S^4$ admits an a.c.s. Now compute

$$\begin{aligned}(\chi - \tau)(M_2 \# a_2 S^4 \times S^4) &= \chi(M_2) + 2a_2 - \tau(M_2) \\ &\equiv 2a_2 \mod 4\,.\end{aligned}$$

By the necessity of condition (b)(i) we conclude $a_1 \equiv a_2 \equiv 0 \mod 2$. □

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