2. The complete ideal symmetric solution of the Tarry-Escott problem of degree four

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$$\sum_{i=1}^{k+1} a_i^r = \sum_{i=1}^{k+1} b_i^r, \qquad r = 1, 2, \ldots, k, k+2. \tag{5}$$

This follows from a theorem given by Gloden [7, p. 24]. Symmetric ideal solutions cannot be used effectively for this purpose as the solutions obtained by applying this theorem hold trivially either for all odd or for all even values of r according as k is odd or even.

In this paper, we will obtain the complete ideal symmetric solution of the Tarry-Escott problem of degree four as well as a parametric ideal non-symmetric solution of this problem. We shall use the non-symmetric solution to obtain a parametric solution of the system of equations

$$\sum_{i=1}^{5} a_i^r = \sum_{i=1}^{5} b_i^r, \qquad r = 1, 2, 3, 4, 6. \tag{6}$$

Parametric solutions of the system of equations (6) have not been obtained earlier.

2. The complete ideal symmetric solution of the Tarry-Escott problem of degree four

To obtain the complete ideal symmetric solution of degree four, we have to obtain a solution of the system of equations

$$\sum_{i=1}^{5} a_i^r = \sum_{i=1}^{5} b_i^r, \qquad r = 1, 2, 3, 4, \tag{7}$$

where $a_i = -b_i$, $i = 1, 2, \ldots, 5$. The four equations of the system (7) now reduce to the following two equations:

$$a_1 + a_2 + a_3 + a_4 + a_5 = 0 \tag{8}$$

and

$$a_1^3 + a_2^3 + a_3^3 + a_4^3 + a_5^3 = 0\,. \tag{9}$$

Thus, to obtain the complete symmetric solution, in reduced form, of the diophantine system (7), we must obtain the complete solution in integers of the equations (8) and (9) such that $\gcd(a_1, a_2, a_3, a_4, a_5) = 1$.

The equations (8) and (9) have trivial solutions in which one of the five integers is zero while the remaining four integers form two pairs, the sum of the integers in each pair being zero, as for example, $(x_1, x_2, -x_1, -x_2, 0)$.

Moreover, it is readily seen that if any solution of (8) and (9) is such that one of the five integers x_i is zero, or the sum of any two of the five integers x_i is zero, then the solution must be trivial. Further, trivial solutions of equations (8) and (9) lead to trivial symmetric solutions of (7).

We will now find the complete non-trivial solution of equations (8) and (9) such that $\gcd(a_1, a_2, a_3, a_4, a_5) = 1$. Let x_i, $i = 1, 2, \ldots, 5$ be any such non-trivial solution of (8) and (9) so that $\gcd(x_1, x_2, x_3, x_4, x_5) = 1$ and the x_i satisfy the equations

$$x_1 + x_2 + x_3 + x_4 + x_5 = 0\,, \tag{10}$$

and

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = 0\,. \tag{11}$$

As our solution is assumed to be non-trivial, we must have $x_1 \neq 0$, $x_2 \neq 0$, $(x_2 + x_3) \neq 0$ and $(x_1 + x_4) \neq 0$ and, accordingly, there must exist non-zero integers p, q, r and s such that

$$px_1 = q(x_2 + x_3)\,, \tag{12}$$

and

$$rx_2 = s(x_1 + x_4)\,. \tag{13}$$

Solving the linear equations (10), (12) and (13), we get

$$\begin{aligned} x_3 &= (px_1 - qx_2)/q\,, \\ x_4 &= (rx_2 - sx_1)/s\,, \\ x_5 &= -(psx_1 + qrx_2)/(qs)\,. \end{aligned} \tag{14}$$

Substituting these values of x_3, x_4 and x_5 in equation (11), we get, on simplification,

$$-3x_1x_2\big[\{(p^2s(r+s) - q^2rs)\}x_1 + \{pq(r^2 - s^2) + q^2r^2\}x_2\big]/(q^2s^2) = 0\,. \tag{15}$$

As $x_1x_2 \neq 0$, it follows from (15) that

$$\begin{aligned} x_1 &= \rho^{-1}\{pq(r^2 - s^2) + q^2r^2\}\,, \\ x_2 &= -\rho^{-1}\{(p^2s(r+s) - q^2rs)\}\,, \end{aligned} \tag{16}$$

where ρ is some rational number. Substituting these values of x_1, x_2 in (14), we get

$$\begin{aligned} x_3 &= \rho^{-1}\{p^2r(r+s) + pqr^2 - q^2rs\}\,, \\ x_4 &= -\rho^{-1}\{p^2r(r+s) + pq(r^2 - s^2)\}\,, \\ x_5 &= \rho^{-1}\{p^2s(r+s) - pqr^2 - q^2r^2\}\,. \end{aligned} \tag{17}$$

Thus, a given non-trivial solution x_i, $i = 1, 2, \ldots, 5$ of equations (8) and (9) must be of the type given by (16) and (17) where p, q, r and s are certain integers and, as we assumed $\gcd(x_1, x_2, x_3, x_4, x_5) = 1$, the rational number ρ must be an integer such that it ensures that $\gcd(x_1, x_2, x_3, x_4, x_5) = 1$.

In accordance with the pattern of equations (16) and (17), we now write

$$\begin{aligned} a_1 &= \rho^{-1}\{pq(r^2 - s^2) + q^2r^2\}, \\ a_2 &= -\rho^{-1}\{(p^2s(r+s) - q^2rs)\}, \\ a_3 &= \rho^{-1}\{p^2r(r+s) + pqr^2 - q^2rs\}, \\ a_4 &= -\rho^{-1}\{p^2r(r+s) + pq(r^2 - s^2)\}, \\ a_5 &= \rho^{-1}\{p^2s(r+s) - pqr^2 - q^2r^2\}, \end{aligned} \tag{18}$$

where p, q, r and s are arbitrary integers and ρ is an integer so chosen that $\gcd(a_1, a_2, a_3, a_4, a_5) = 1$. It is now readily verified by direct substitution that a_1, a_2, a_3, a_4, a_5 as defined by (18) satisfy both the equations (8) and (9). It has already been seen that any given non-trivial solution of (8) and (9) is of the type (18), and hence it follows that this is the complete non-trivial solution of equations (8) and (9).

It now follows that the complete ideal symmetric solution of the Tarry-Escott problem of degree four is given in the reduced form by $a_i = -b_i$, $i = 1, 2, \ldots, 5$, where a_1, a_2, a_3, a_4, a_5 are defined by (18) in terms of the arbitrary integer parameters p, q, r and s while ρ is an integer so chosen that $\gcd(a_1, a_2, a_3, a_4, a_5) = 1$. Symmetric ideal solutions that are not in the reduced form may be obtained by the application of Frolov's theorem to the above symmetric ideal solution.

As a numerical example, when $p = 1$, $q = 1$, $r = 2$, $s = 1$, $\rho = 1$, we get, after suitable re-arrangement, the following reduced ideal symmetric solution of the Tarry-Escott problem of degree 4:

$$(-9)^r + (-5)^r + (-1)^r + 7^r + 8^r = (-8)^r + (-7)^r + 1^r + 5^r + 9^r, \quad r = 1, 2, 3, 4.$$

Adding the constant 10 to all the terms, we get the following symmetric solution in positive integers:

$$1^r + 5^r + 9^r + 17^r + 18^r = 2^r + 3^r + 11^r + 15^r + 19^r, \quad r = 1, 2, 3, 4.$$