

4.3 Relating $L_p(s, t; \lambda)$ to some finite sums

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **46 (2000)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **18.04.2024**

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With this result, we can derive a more general power series expansion of $L_p(s, t; \chi)$.

THEOREM 4.7. *Let $t \in \mathbf{C}_p$, $|t|_p \leq 1$, and $s \in \mathfrak{D}$, except $s \neq 1$ if $\chi = 1$. Then for $\alpha \in \mathbf{C}_p$, $|\alpha|_p \leq 1$,*

$$L_p(s, t; \chi) = \sum_{m=0}^{\infty} \binom{-s}{m} q^m (t - \alpha)^m L_p(s + m, \alpha; \chi_m).$$

REMARK. Note that Theorem 4.5 is the case of $\alpha = 0$ here.

Proof. It follows from the Taylor series expansion of $L_p(s, t; \chi)$ in the variable t about α (see Proposition 2.6) that we can write $L_p(s, t; \chi)$ in the form

$$L_p(s, t; \chi) = \sum_{m=0}^{\infty} \beta_m (t - \alpha)^m,$$

where

$$\beta_m = \frac{1}{m!} \frac{\partial^m}{\partial t^m} L_p(s, t; \chi) \Big|_{t=\alpha}.$$

From Lemma 4.6

$$\frac{1}{m!} \frac{\partial^m}{\partial t^m} L_p(s, t; \chi) = \binom{-s}{m} q^m L_p(s + m, t; \chi_m),$$

and so

$$\beta_m = \binom{-s}{m} q^m L_p(s + m, \alpha; \chi_m),$$

completing the proof. $\square$

4.3 RELATING $L_p(s, t; \chi)$ TO SOME FINITE SUMS

From (4) it becomes obvious that the generalized Bernoulli polynomials have a considerable significance in regard to sums of consecutive nonnegative integers, each raised to the same power, itself a nonnegative integer. The following illustrates how this can be extended with the use of $L_p(s, t; \chi)$.

For the character χ , let $F_0 = \text{lcm}(f_\chi, q)$. Then $f_{\chi_n} \mid F_0$ for each $n \in \mathbf{Z}$. Also, let F be a positive multiple of $pq^{-1}F_0$.

THEOREM 4.8. Let $t \in \mathbf{C}_p$, $|t|_p \leq 1$, and $s \in \mathfrak{D}$, except $s \neq 1$ if $\chi = 1$. Then

$$(23) \quad L_p(s, t + F; \chi) - L_p(s, t; \chi) = - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a) \langle a + qt \rangle^{-s}.$$

Proof. Let $t \in \mathbf{C}_p$, $|t|_p \leq 1$, and let $n \in \mathbf{Z}$, $n \geq 1$. Then from (18),

$$L_p(1 - n, t + F; \chi) - L_p(1 - n, t; \chi) = -\frac{1}{n} (b_n(t + F) - b_n(t)).$$

Now, (19) implies

$$\begin{aligned} b_n(t + F) - b_n(t) &= (B_{n, \chi_n}(q(t + F)) - \chi_n(p)p^{n-1}B_{n, \chi_n}(p^{-1}q(t + F))) \\ &\quad - (B_{n, \chi_n}(qt) - \chi_n(p)p^{n-1}B_{n, \chi_n}(p^{-1}qt)) \\ &= (B_{n, \chi_n}(q(t + F)) - B_{n, \chi_n}(qt)) \\ &\quad - \chi_n(p)p^{n-1} (B_{n, \chi_n}(p^{-1}q(t + F)) - B_{n, \chi_n}(p^{-1}qt)). \end{aligned}$$

Thus, by (4), we can write

$$\begin{aligned} b_n(t + F) - b_n(t) &= n \sum_{a=1}^{qF} \chi_n(a)(a + qt)^{n-1} - n \chi_n(p)p^{n-1} \sum_{a=1}^{p^{-1}qF} \chi_n(a)(a + p^{-1}qt)^{n-1} \\ &= n \sum_{a=1}^{qF} \chi_n(a)(a + qt)^{n-1} - n \sum_{\substack{a=1 \\ p|a}}^{qF} \chi_n(a)(a + qt)^{n-1}. \end{aligned}$$

Therefore,

$$L_p(1 - n, t + F; \chi) - L_p(1 - n, t; \chi) = - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_n(a)(a + qt)^{n-1}.$$

Now, $\chi_n = \chi_1 \omega^{-(n-1)}$, so that

$$\begin{aligned} \chi_n(a)(a + qt)^{n-1} &= \chi_1(a) \omega^{-(n-1)}(a)(a + qt)^{n-1} \\ &= \chi_1(a) \langle a + qt \rangle^{n-1}. \end{aligned}$$

Thus

$$L_p(1 - n, t + F; \chi) - L_p(1 - n, t; \chi) = - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a) \langle a + qt \rangle^{n-1},$$

and (23) holds for all $s = 1 - n$, where $n \in \mathbf{Z}$, $n \geq 1$. Therefore, since the negative integers have 0 as a limit point, Lemma 2.5 implies that Theorem 4.8 holds for all s in any neighborhood about 0 common to the domains of the functions on either side of (23).

It is obvious that the domains, in the variable s , of the functions on the left of (23) contain $\mathfrak{D}$, except $s \neq 1$ when $\chi = 1$. Consider now the function

$$- \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a) \langle a + qt \rangle^{-s} = - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a) \langle a + qt \rangle^{-1} \langle a + qt \rangle^{1-s}.$$

Since it consists of a finite sum of functions of the form $\langle a + qt \rangle^{1-s}$, where $a \in \mathbf{Z}$, $(a, p) = 1$, we need only show that each such function is analytic on $\mathfrak{D}$, and the proof will be complete.

The quantity $\langle a + qt \rangle^{1-s}$ can be written as

$$\langle a + qt \rangle^{1-s} = \exp((1-s) \log \langle a + qt \rangle),$$

and by (9), the Taylor series expansion of the exponential function,

$$\langle a + qt \rangle^{1-s} = \sum_{m=0}^{\infty} \frac{1}{m!} (1-s)^m (\log \langle a + qt \rangle)^m.$$

Since $\langle a + qt \rangle \equiv 1 \pmod{q\mathfrak{o}}$ for $a \in \mathbf{Z}$, $(a, p) = 1$, and $t \in \mathbf{C}_p$, $|t|_p \leq 1$, we must also have $\log \langle a + qt \rangle \equiv 0 \pmod{q\mathfrak{o}}$ for such a and t . Thus

$$\left| \frac{1}{m!} (1-s)^m (\log \langle a + qt \rangle)^m \right|_p \leq \left| \frac{1}{m!} q^m (s-1)^m \right|_p$$

for all m . By (8) we can write

$$\begin{aligned} \left| \frac{1}{m!} q^m (s-1)^m \right|_p &\leq \left| p^{-m/(p-1)} q^m (s-1)^m \right|_p \\ &= \left| p^{-1/(p-1)} q (s-1) \right|_p^m. \end{aligned}$$

Thus if

$$\left| p^{-1/(p-1)} q (s-1) \right|_p < 1,$$

then

$$\left| \frac{1}{m!} (1-s)^m (\log \langle a + qt \rangle)^m \right|_p \rightarrow 0$$

as $m \rightarrow \infty$. So whenever $|s-1|_p < |p|_p^{1/(p-1)} |q|_p^{-1}$, meaning that $s \in \mathfrak{D}$, we have convergence for the power series. Therefore, the functions on either side of (23) have domains that contain $\mathfrak{D}$, except possibly for $s = 1$ when $\chi = 1$, and the theorem must hold. $\square$

COROLLARY 4.9. *Let $s \in \mathfrak{D}$, except $s \neq 1$ if $\chi = 1$. Then*

$$L_p(s, F; \chi) = L_p(s; \chi) - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a) \langle a \rangle^{-s}.$$

Proof. This follows from Theorem 4.8 since $L_p(s, 0; \chi) = L_p(s; \chi)$ for any character χ . $\square$

We shall now consider how Corollary 4.9 can be utilized to derive a collection of congruences related to the generalized Bernoulli polynomials. Let Δ_c denote the forward difference operator, $\Delta_c x_n = x_{n+c} - x_n$. Repeated application of this operator can be expressed in the form

$$\Delta_c^k x_n = \sum_{m=0}^k \binom{k}{m} (-1)^{k-m} x_{n+mc}.$$

Recall that $F_0 = \text{lcm}(f_\chi, q)$. For $n \in \mathbf{Z}$, $n \geq 1$, denote

$$\beta_{n,\chi}(t) = -\frac{1}{n} \left(B_{n,\chi_n}(qt) - \chi_n(p) p^{n-1} B_{n,\chi_n}(p^{-1}qt) \right).$$

This is the polynomial structure that we utilized with respect to generalizing the p -adic L -functions. We will incorporate this structure in an extension of the Kummer congruences, but the results that we derive will be without restriction on either χ or p .

THEOREM 4.10. *Let n , c , and k be positive integers, and let $\tau \in \mathbf{Z}_p$ such that $|\tau|_p \leq |pq^{-1}F_0|_p$. Then the quantity $q^{-k} \Delta_c^k \beta_{n,\chi}(\tau) - q^{-k} \Delta_c^k \beta_{n,\chi}(0) \in \mathbf{Z}_p[\chi]$, and, modulo $q\mathbf{Z}_p[\chi]$, is independent of n .*

Proof. Since Δ_c is a linear operator, Corollary 4.9 implies that

$$\Delta_c^k L_p(1-n, F; \chi) = \Delta_c^k L_p(1-n; \chi) - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a) \Delta_c^k \langle a \rangle^{n-1},$$

where F is a positive multiple of $pq^{-1}F_0$. Thus

$$\Delta_c^k \beta_{n,\chi}(F) - \Delta_c^k \beta_{n,\chi}(0) = - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a) \langle a \rangle^{-1} \Delta_c^k \langle a \rangle^n.$$

Note that

$$(24) \quad \Delta_c^k \langle a \rangle^n = \sum_{m=0}^k \binom{k}{m} (-1)^{k-m} \langle a \rangle^{n+mc} = \langle a \rangle^n (\langle a \rangle^c - 1)^k.$$

Now, $\langle a \rangle \equiv 1 \pmod{q\mathbf{Z}_p}$, which implies that $\langle a \rangle^c \equiv 1 \pmod{q\mathbf{Z}_p}$, and thus

$$\Delta_c^k \langle a \rangle^n \equiv 0 \pmod{q^k \mathbf{Z}_p}.$$

Therefore

$$\Delta_c^k \beta_{n,\chi}(F) - \Delta_c^k \beta_{n,\chi}(0) \equiv 0 \pmod{q^k \mathbf{Z}_p[\chi]},$$

and so $q^{-k} \Delta_c^k \beta_{n,\chi}(F) - q^{-k} \Delta_c^k \beta_{n,\chi}(0) \in \mathbf{Z}_p[\chi]$. Also, since $\langle a \rangle^n \equiv 1 \pmod{q\mathbf{Z}_p}$,

$$(25) \quad q^{-k} \Delta_c^k \beta_{n,\chi}(F) - q^{-k} \Delta_c^k \beta_{n,\chi}(0) = - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a) \langle a \rangle^{n-1} \left(\frac{\langle a \rangle^c - 1}{q} \right)^k$$

implies that the value of $q^{-k} \Delta_c^k \beta_{n,\chi}(F) - q^{-k} \Delta_c^k \beta_{n,\chi}(0)$ modulo $q\mathbf{Z}_p[\chi]$ is independent of n .

Let $\tau \in pq^{-1}F_0\mathbf{Z}_p$. Since the set of positive integers in $pq^{-1}F_0\mathbf{Z}$ is dense in $pq^{-1}F_0\mathbf{Z}_p$, there exists a sequence $\{\tau_i\}_{i=1}^\infty$ in $pq^{-1}F_0\mathbf{Z}$, with $\tau_i > 0$ for each i , such that $\tau_i \rightarrow \tau$. Now, $\beta_{n,\chi}(t)$ is a polynomial, which implies that $\beta_{n,\chi}(\tau_i) \rightarrow \beta_{n,\chi}(\tau)$. Therefore

$$\lim_{i \rightarrow \infty} (\Delta_c^k \beta_{n,\chi}(\tau_i) - \Delta_c^k \beta_{n,\chi}(0)) = \Delta_c^k \beta_{n,\chi}(\tau) - \Delta_c^k \beta_{n,\chi}(0).$$

The left side of this equality is 0 modulo $q^k \mathbf{Z}_p[\chi]$, which implies that

$$\Delta_c^k \beta_{n,\chi}(\tau) - \Delta_c^k \beta_{n,\chi}(0) \equiv 0 \pmod{q^k \mathbf{Z}_p[\chi]},$$

and so $q^{-k} \Delta_c^k \beta_{n,\chi}(\tau) - q^{-k} \Delta_c^k \beta_{n,\chi}(0) \in \mathbf{Z}_p[\chi]$. Furthermore, for n' a positive integer,

$$\begin{aligned} & \lim_{i \rightarrow \infty} ((q^{-k} \Delta_c^k \beta_{n,\chi}(\tau_i) - q^{-k} \Delta_c^k \beta_{n,\chi}(0)) - (q^{-k} \Delta_c^k \beta_{n',\chi}(\tau_i) - q^{-k} \Delta_c^k \beta_{n',\chi}(0))) \\ &= ((q^{-k} \Delta_c^k \beta_{n,\chi}(\tau) - q^{-k} \Delta_c^k \beta_{n,\chi}(0)) - (q^{-k} \Delta_c^k \beta_{n',\chi}(\tau) - q^{-k} \Delta_c^k \beta_{n',\chi}(0))). \end{aligned}$$

Since $\tau_i \in pq^{-1}F_0\mathbf{Z}$ for each i , the quantity on the left must also be 0 modulo $q\mathbf{Z}_p[\chi]$. Therefore the value of $q^{-k} \Delta_c^k \beta_{n,\chi}(\tau) - q^{-k} \Delta_c^k \beta_{n,\chi}(0)$ modulo $q\mathbf{Z}_p[\chi]$ is independent of n . $\square$

THEOREM 4.11. Let n , c , k , and k' be positive integers with $k \equiv k' \pmod{p-1}$, and let $\tau \in \mathbf{Z}_p$ such that $|\tau|_p \leq |pq^{-1}F_0|_p$. Then

$$\begin{aligned} q^{-k}\Delta_c^k\beta_{n,\chi}(\tau) - q^{-k}\Delta_c^k\beta_{n,\chi}(0) \\ \equiv q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(\tau) - q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(0) \pmod{p\mathbf{Z}_p[\chi]}. \end{aligned}$$

Proof. Let k and k' be positive integers such that $k \equiv k' \pmod{p-1}$. Without loss of generality, we can assume that $k \geq k'$. From (25),

$$\begin{aligned} & (q^{-k}\Delta_c^k\beta_{n,\chi}(F) - q^{-k}\Delta_c^k\beta_{n,\chi}(0)) - (q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(F) - q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(0)) \\ &= - \sum_{\substack{a=1 \\ (a,p)=1}}^{q^F} \chi_1(a)\langle a \rangle^{n-1} \left(\frac{\langle a \rangle^c - 1}{q} \right)^k + \sum_{\substack{a=1 \\ (a,p)=1}}^{q^F} \chi_1(a)\langle a \rangle^{n-1} \left(\frac{\langle a \rangle^c - 1}{q} \right)^{k'} \\ &= - \sum_{\substack{a=1 \\ (a,p)=1}}^{q^F} \chi_1(a)\langle a \rangle^{n-1} \left(\frac{\langle a \rangle^c - 1}{q} \right)^{k'} \left(\left(\frac{\langle a \rangle^c - 1}{q} \right)^{k-k'} - 1 \right), \end{aligned}$$

where F is a positive multiple of $pq^{-1}F_0$. If a is such that

$$\langle a \rangle^c - 1 \not\equiv 0 \pmod{pq\mathbf{Z}_p},$$

then

$$\left(\frac{\langle a \rangle^c - 1}{q} \right)^{k-k'} - 1 \equiv 0 \pmod{pq\mathbf{Z}_p},$$

since $k - k' \equiv 0 \pmod{p-1}$. Thus

$$\begin{aligned} q^{-k}\Delta_c^k\beta_{n,\chi}(F) - q^{-k}\Delta_c^k\beta_{n,\chi}(0) \\ \equiv q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(F) - q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(0) \pmod{p\mathbf{Z}_p[\chi]}. \end{aligned}$$

Now let $\tau \in pq^{-1}F_0\mathbf{Z}_p$. Then there exists a sequence $\{\tau_i\}_{i=1}^\infty$ in $pq^{-1}F_0\mathbf{Z}$, with $\tau_i > 0$ for each i , such that $\tau_i \rightarrow \tau$. Consider

$$\begin{aligned} & \lim_{i \rightarrow \infty} ((q^{-k}\Delta_c^k\beta_{n,\chi}(\tau_i) - q^{-k}\Delta_c^k\beta_{n,\chi}(0)) - (q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(\tau_i) - q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(0))) \\ &= (q^{-k}\Delta_c^k\beta_{n,\chi}(\tau) - q^{-k}\Delta_c^k\beta_{n,\chi}(0)) - (q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(\tau) - q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(0)). \end{aligned}$$

Since the left side of this equality must be 0 modulo $p\mathbf{Z}_p[\chi]$, the theorem must hold. $\square$

THEOREM 4.12. Let n , c , and k be positive integers, and let $\tau \in \mathbf{Z}_p$ such that $|\tau|_p \leq |pq^{-1}F_0|_p$. Then the quantity

$$\binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(\tau) - \binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(0) \in \mathbf{Z}_p[\chi],$$

and, modulo $q\mathbf{Z}_p[\chi]$, is independent of n .

Proof. We are once again working with a linear operator, so Corollary 4.9 implies that

$$\binom{q^{-1}\Delta_c}{k} L_p(1-n, F; \chi) = \binom{q^{-1}\Delta_c}{k} L_p(1-n; \chi) - \sum_{\substack{a=1 \\ (a,p)=1}}^{q^F} \chi_1(a) \binom{q^{-1}\Delta_c}{k} \langle a \rangle^{n-1},$$

where F is a positive multiple of $pq^{-1}F_0$. Then

$$\binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(F) - \binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(0) = - \sum_{\substack{a=1 \\ (a,p)=1}}^{q^F} \chi_1(a) \langle a \rangle^{-1} \binom{q^{-1}\Delta_c}{k} \langle a \rangle^n.$$

Utilizing (15), we can write

$$\begin{aligned} \binom{q^{-1}\Delta_c}{k} \langle a \rangle^n &= \frac{1}{k!} \sum_{m=0}^k s(k, m) q^{-m} \Delta_c^m \langle a \rangle^n \\ &= \frac{1}{k!} \sum_{m=0}^k s(k, m) q^{-m} \langle a \rangle^n (\langle a \rangle^c - 1)^m, \end{aligned}$$

which follows from (24). This can then be rewritten as

$$\binom{q^{-1}\Delta_c}{k} \langle a \rangle^n = \langle a \rangle^n \binom{q^{-1}(\langle a \rangle^c - 1)}{k}.$$

Since $q^{-1}(\langle a \rangle^c - 1) \in \mathbf{Z}_p$ for each $a \in \mathbf{Z}$ with $(a, p) = 1$, we see that

$$\langle a \rangle^n \binom{q^{-1}(\langle a \rangle^c - 1)}{k} \in \mathbf{Z}_p.$$

This then implies that

$$\binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(F) - \binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(0) \in \mathbf{Z}_p[\chi].$$

Furthermore, since $\langle a \rangle^n \equiv 1 \pmod{q\mathbf{Z}_p}$, the value of this quantity modulo $q\mathbf{Z}_p[\chi]$ is independent of n .

Now let $\tau \in pq^{-1}F_0\mathbf{Z}_p$, and let $\{\tau_i\}_{i=1}^\infty$ be a sequence in $pq^{-1}F_0\mathbf{Z}$, with $\tau_i > 0$ for each i , such that $\tau_i \rightarrow \tau$. We are working with polynomials, so that

$$\begin{aligned} \lim_{i \rightarrow \infty} \left(\binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(\tau_i) - \binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(0) \right) \\ = \binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(\tau) - \binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(0), \end{aligned}$$

which must be in $\mathbf{Z}_p[\chi]$ since the limit of any sequence in $\mathbf{Z}_p[\chi]$ must also be in $\mathbf{Z}_p[\chi]$. Now let n' be a positive integer, and consider

$$\begin{aligned} \lim_{i \rightarrow \infty} \left(\left(\binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(\tau_i) - \binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(0) \right) - \left(\binom{q^{-1}\Delta_c}{k} \beta_{n',\chi}(\tau_i) - \binom{q^{-1}\Delta_c}{k} \beta_{n',\chi}(0) \right) \right) \\ = \left(\left(\binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(\tau) - \binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(0) \right) - \left(\binom{q^{-1}\Delta_c}{k} \beta_{n',\chi}(\tau) - \binom{q^{-1}\Delta_c}{k} \beta_{n',\chi}(0) \right) \right). \end{aligned}$$

The quantity on the left must be 0 modulo $q\mathbf{Z}_p[\chi]$, which implies that the value of

$$\binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(\tau) - \binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(0)$$

modulo $q\mathbf{Z}_p[\chi]$ is independent of n . $\square$

4.4 GENERALIZED BERNOULLI POWER SERIES

In [9] we find a definition of ordinary Bernoulli numbers of negative index, B_{-n} , where $n \in \mathbf{Z}$, $n \geq 1$, in the field $\mathbf{Q}_p$, given by

$$(26) \quad B_{-n} = \lim_{k \rightarrow \infty} B_{\phi(p^k) - n},$$

where the limit is taken in a p -adic sense. Note that $\phi(p^k) \rightarrow 0$ in $\mathbf{Z}_p$ as $k \rightarrow \infty$. Since $|B_m|_p$ is bounded for all $m \in \mathbf{Z}$, $m \geq 0$, we must have

$$\begin{aligned} B_{-n} &= \lim_{k \rightarrow \infty} \left(1 - p^{\phi(p^k) - n - 1} \right) B_{\phi(p^k) - n} \\ &= \lim_{k \rightarrow \infty} -(\phi(p^k) - n) L_p(1 - (\phi(p^k) - n); \omega^{-n}) \\ &= n L_p(n + 1; \omega^{-n}). \end{aligned}$$

implying that the limit exists and can be described in familiar terms.

Recall that $B_m = 0$ for any odd $m \in \mathbf{Z}$, $m \geq 3$. Thus (26) implies that $B_{-n} = 0$ for any odd $n \in \mathbf{Z}$, $n \geq 1$. Furthermore, we have the following: