2.5 p-ADIC FUNCTIONS

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LEMMA 2.3. *Let $\tau \in \mathbf{C}_p$. In the field $\mathbf{Q}_p(\chi, \tau)$, for all $n \in \mathbf{Z}$, $n \geq 0$,*

$$B_{n,\chi_n}(\tau) = \lim_{h\to\infty} \frac{1}{p^h f_\chi} \sum_{a=1}^{p^h f_\chi} \chi_n(a)(a+\tau)^n .$$

Proof. By applying Lemma 2.2 to (4), we obtain

$$B_{n,\chi} = \lim_{h\to\infty} \frac{1}{p^h f_\chi} \sum_{a=1}^{p^h f_\chi} \chi(a)a^n .$$

Therefore, by (2),

$$\begin{aligned} B_{n,\chi_n}(\tau) &= \sum_{m=0}^{n} \binom{n}{m} \tau^{n-m} \lim_{h\to\infty} \frac{1}{p^h f_{\chi_n}} \sum_{a=1}^{p^h f_{\chi_n}} \chi_n(a)a^m \\ &= \lim_{h\to\infty} \frac{1}{p^h f_{\chi_n}} \sum_{a=1}^{p^h f_{\chi_n}} \chi_n(a) \sum_{m=0}^{n} \binom{n}{m} \tau^{n-m} a^m . \end{aligned}$$

Since f_χ and f_{χ_n} differ by a factor that is a power of p, we must have

$$B_{n,\chi_n}(\tau) = \lim_{h\to\infty} \frac{1}{p^h f_\chi} \sum_{a=1}^{p^h f_\chi} \chi_n(a)(a+\tau)^n ,$$

and the proof is complete. □

2.5 p-ADIC FUNCTIONS

Let K be an extension of $\mathbf{Q}_p$ contained in $\mathbf{C}_p$. An infinite series $\sum_{n=0}^{\infty} a_n$, $a_n \in K$, converges in K if and only if $|a_n|_p \to 0$ as $n \to \infty$. Let $K[[x]]$ be the algebra of formal power series in x. Then it follows that a power series

$$A(x) = \sum_{n=0}^{\infty} a_n x^n ,$$

in $K[[x]]$, converges at $x = \xi$, $\xi \in \mathbf{C}_p$, if and only if $|a_n \xi^n|_p \to 0$ as $n \to \infty$. Therefore whenever a power series $A(x)$ converges at some $\xi_0 \in \mathbf{C}_p$, then it must converge at all $\xi \in \mathbf{C}_p$ such that $|\xi|_p \leq |\xi_0|_p$. The following result, for double series in K, can be found in [8].

PROPOSITION 2.4. *Let $b_{n,m} \in K$, and suppose that for each $\epsilon > 0$ there exists $N \in \mathbf{Z}$, depending on ϵ, such that if* $\max\{n,m\} \geq N$, *then $|b_{n,m}|_p \leq \epsilon$. Then both series*

$$\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} b_{n,m}\right) \quad \textit{and} \quad \sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} b_{n,m}\right)$$

converge, and their sums are equal.

There are two power series that we wish to make note of in particular. First we define the p-adic exponential function, $\exp(x)$, in $\mathbf{Q}_p[[x]]$, by

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n. \tag{9}$$

From (8) we can conclude that this power series converges in $\{x \in \mathbf{C}_p : |x|_p < p^{-1/(p-1)}\}$. The p-adic logarithm function, $\log(x)$, in $\mathbf{Q}_p[[x]]$, is defined by

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n, \tag{10}$$

the power series converging in the domain $\{x \in \mathbf{C}_p : |x|_p < 1\}$. For $|x|_p < p^{-1/(p-1)}$, we have $\log(\exp(x)) = x$ and $\exp(\log(1+x)) = 1+x$.

The following property is a uniqueness property for power series, found in [13].

LEMMA 2.5. *Let $A(x), B(x) \in K[[x]]$, such that each converges in a neighborhood of 0 in $\mathbf{C}_p$. If $A(\xi_n) = B(\xi_n)$ for a sequence $\{\xi_n\}_{n=0}^{\infty}$, $\xi_n \neq 0$, in $\mathbf{C}_p$, such that $\xi_n \to 0$, then $A(x) = B(x)$.*

Let U be an open subset of $\mathbf{C}_p$, contained in the domain of the p-adic function f. We say that f is differentiable at $x \in U$ if the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists. If this limit exists for each $x \in U$, then we say that f is differentiable in U.

The relationship between the derivatives of a function and its power series expansion is given in the following result, found in [8].

PROPOSITION 2.6. *Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with coefficients in $\mathbf{C}_p$, and suppose that*

$$f(x) = \sum_{n=0}^{\infty} a_n (x-\alpha)^n$$

converges on some closed ball B in $\mathbf{C}_p$. Then

i) *For each $x \in B$, the k^{th} derivative $f^{(k)}(x)$ exists, and is given by*

$$f^{(k)}(x) = k! \sum_{n=k}^{\infty} \binom{n}{k} a_n (x-\alpha)^{n-k},$$

and we have

$$a_k = \frac{1}{k!} f^{(k)}(\alpha).$$

ii) *Let $\beta \in B$. Then there exists a series $\sum_{n=0}^{\infty} b_n x^n$ such that*

$$f(x) = \sum_{n=0}^{\infty} b_n (x-\beta)^n$$

for any $x \in B$. Both series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ have the same region of convergence.

Now let K be a finite extension of $\mathbf{Q}_p$. For $A(x) \in K[[x]]$, $A(x) = \sum_{n=0}^{\infty} a_n x^n$, where $a_n \in K$, define

$$\|A\| = \sup_n |a_n|_p .$$

Let $P_K = \{A(x) \in K[[x]] : \|A\| < \infty\}$. Then $\|\cdot\|$ defines a norm on P_K, and so $K[x] \subset P_K \subset K[[x]]$. Furthermore P_K is complete in this norm.

Let $\{b_n\}_{n=0}^{\infty}$ be a sequence of elements of K, and let the sequence $\{c_n\}_{n=0}^{\infty}$ be defined by

$$c_n = \sum_{m=0}^{n} \binom{n}{m} (-1)^{n-m} b_m \tag{11}$$

for each $n \in \mathbf{Z}$, $n \geq 0$. Then $c_n \in K$ for each $n \geq 0$. Note that (11) implies that these sequences must satisfy

$$\sum_{n=0}^{\infty} c_n \frac{t^n}{n!} = e^{-t} \sum_{n=0}^{\infty} b_n \frac{t^n}{n!} .$$

This implies that

$$\sum_{n=0}^{\infty} b_n \frac{t^n}{n!} = e^t \sum_{n=0}^{\infty} c_n \frac{t^n}{n!},$$

and so we have the relationship

$$b_n = \sum_{m=0}^{n} \binom{n}{m} c_m \tag{12}$$

for each $n \in \mathbf{Z}$, $n \geq 0$. We can reverse this process to derive (11) given (12). Thus (11) and (12) must be equivalent. The following relate to sequences that satisfy (11) and (12), and are found in [13].

THEOREM 2.7. *Let $\{b_n\}_{n=0}^{\infty}$ and $\{c_n\}_{n=0}^{\infty}$ be defined as in the above relation. Let $\rho \in \mathbf{R}$ such that $0 < \rho < |p|_p^{1/(p-1)}$. If $|c_n|_p \leq C\rho^n$ for all $n \geq 0$, where $C > 0$, then there exists a unique power series $A(x) \in P_K$ such that $A(x)$ converges at every $\xi \in \mathbf{C}_p$ with $|\xi|_p < |p|_p^{1/(p-1)}\rho^{-1}$, and $A(n) = b_n$ for every $n \geq 0$.*

COROLLARY 2.8. *Let $A(x)$ be the power series from the theorem. Then for each $\xi \in \mathbf{C}_p$ such that $|\xi|_p < |p|_p^{1/(p-1)}\rho^{-1}$, we have*

$$A(\xi) = \sum_{n=0}^{\infty} c_n \binom{\xi}{n}.$$

Theorem 2.7 can be applied to the sequence $\{b_n\}_{n=0}^{\infty}$ in $K = \mathbf{Q}_p(\chi)$, where

$$b_n = \left(1 - \chi_n(p)p^{n-1}\right) B_{n,\chi_n},$$

in order to obtain a power series $A_\chi(s)$ satisfying $A_\chi(n) = b_n$, and converging on the domain $\{s \in \mathbf{C}_p : |s|_p < |p|_p^{1/(p-1)}|q|_p^{-1}\}$. (Since $|p|_p^{1/(p-1)}|q|_p^{-1} > 1$ and $|n|_p \leq 1$ for each $n \in \mathbf{Z}$, all of $\mathbf{Z}$ is contained in this domain.) From this a p-adic function, $L_p(s;\chi)$, can be derived that interpolates the values

$$L_p(1-n;\chi) = -\frac{1}{n} b_n,$$

and which converges in $\{s \in \mathbf{C}_p : |s-1|_p < |p|_p^{1/(p-1)}|q|_p^{-1}\}$, except $s \neq 1$ if $\chi = 1$. Note that if χ is odd, then χ_n is even when n is odd, and χ_n is odd when n is even. Thus the quantity $(1-\chi_n(p)p^{n-1})B_{n,\chi_n} = 0$ for all $n \in \mathbf{Z}$, $n \geq 1$, as we saw from the properties of generalized Bernoulli numbers. Therefore $L_p(s;\chi)$ vanishes on a sequence such as $\{-p^m\}_{m=0}^{\infty}$, which has 0 as a limit point, implying that for such χ we must have $L_p(s;\chi) \equiv 0$.