# 2.3 Dirichlet L-functions

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## 2.3 DIRICHLET $L$-FUNCTIONS

For $\chi$ a Dirichlet character with conductor $f_\chi$, the Dirichlet $L$-function for $\chi$ is defined by

$$L(s;\chi) = \sum_{b=1}^{\infty} \frac{\chi(b)}{b^s},$$

for $s \in \mathbf{C}$ such that $\Re(s) > 1$. Note that $L(s;\chi)$ can be continued analytically to all of $\mathbf{C}$, except for a pole of order 1 at $s = 1$ when $\chi = 1$.

Let $\tau(\chi)$ be a Gauss sum,

$$\tau(\chi) = \sum_{a=1}^{f_\chi} \chi(a) e^{2\pi i a/f_\chi},$$

where $i^2 = -1$, and let

$$\delta_\chi = \begin{cases} 0, & \text{if } \chi(-1) = 1 \\ 1, & \text{if } \chi(-1) = -1. \end{cases}$$

Then $L(s;\chi)$ satisfies the functional equation

$$\left(\frac{f_\chi}{\pi}\right)^{s/2} \Gamma\left(\frac{s+\delta_\chi}{2}\right) L(s;\chi) = W_\chi \left(\frac{f_\chi}{\pi}\right)^{(1-s)/2} \Gamma\left(\frac{1-s+\delta_\chi}{2}\right) L(1-s;\overline{\chi}), \tag{7}$$

where $\Gamma(s)$ is the gamma function, and $W_\chi = \frac{\tau(\chi)}{i^{\delta_\chi} \cdot \sqrt{f_\chi}}$, having the property that $|W_\chi| = 1$. Since $\Gamma(s)$ has simple poles at the negative integers, $L(s;\chi)$ must be zero for $s = 1 - n$, where $n \in \mathbf{Z}$, $n \geq 1$, such that $n \not\equiv \delta_\chi \pmod 2$, except when $\chi = 1$ and $n = 1$. $L(s;\chi)$ can also be described by means of the Euler product $L(s;\chi) = \prod_{p \text{ prime}} \left(1 - \chi(p) p^{-s}\right)^{-1}$, for $s \in \mathbf{C}$ such that $\Re(s) > 1$. Thus $L(s;\chi) \neq 0$ in this domain.

The generalized Bernoulli numbers, $B_{n,\chi}$, and the Dirichlet $L$-function, $L(s;\chi)$, share the following relationship, a proof of this being found in [13]:

THEOREM 2.1. *Let $\chi$ be a Dirichlet character, and let $n \in \mathbf{Z}$, $n \geq 1$. Then $L(1-n;\chi) = -\frac{1}{n} B_{n,\chi}$.*

Thus we have a way to express certain values of a function defined in terms of an infinite sum as quantities that can be found by a finite process.