7.4 TOUGHER EXAMPLES

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of a cycle is the regular tree $\mathcal{T}$ of degree 2, and circuits in C correspond bijectively to paths in $\mathcal{T}$ from $\star$ to any vertex at distance a multiple of k. We thus have

$$F_C(u,t) = \sum_{\zeta:\zeta^k=1} H(u,\zeta,t)$$

where the sum runs over all kth roots of unity and $d=2$ in H.

We consider next the following graphs: take a d-regular tree and fix a vertex $\star$. At $\star$, delete e vertices and replace them by e loops. Then clearly

$$F(0,t) = \frac{1+t}{1-(e-1)t},$$

as all the non-backtracking paths are constrained to the e loops. Using (2.3), we obtain after simplifications

$$G(t) = \frac{2(d-1)}{d+e-2-2e(d-1)t+(d-e)\sqrt{1-4(d-1)t^2}}. \tag{7.2}$$

The radius of convergence of G is

$$\min\left\{\frac{1}{2\sqrt{d-1}}, \frac{e-1}{d+e^2-2e}\right\}.$$

7.4 Tougher examples

In this subsection we outline the computations of F and G for more complicated graphs. They are only provided as examples and are logically independent from the remainder of the paper. The arguments will therefore be somewhat condensed.

First take for $\mathcal{X}$ the Cayley graph of $\Gamma = (\mathbf{Z}/2\mathbf{Z}) \times \mathbf{Z}$ with generators $(0,-1) = `\downarrow'$, $(0,1) = `\uparrow'$ and $(1,0) = `\leftrightarrow'$. Geometrically, $\mathcal{X}$ is a doubly-infinite two-poled ladder.

In Subsection 7.3 we computed

$$F_{\mathbf{Z}}(u,t) = \frac{1-(1-u)^2t^2}{\sqrt{(1+(1-u^2)t^2)^2-4t^2}},$$

the growth of circuits restricted to one pole of the ladder. A circuit in $\mathcal{X}$ is a circuit in $\mathbf{Z}$, before and after each step ($\uparrow$ or $\downarrow$) of which we may switch to the other pole (with a $\leftrightarrow$) as many times as we wish, subject to the condition that the circuit finish at the same pole as it started. This last condition is expressed by the fact that the series we obtain must have only coefficients of even degree in t. Thus, letting $\mathrm{even}(f) = \frac{f(t)+f(-t)}{2}$, we have

$$G(t) = \text{even}\Big(\frac{1}{1-t}F_{\mathbf{Z}}\Big(1, \frac{t}{1-t}\Big)\Big)\,;$$

it is then simple to obtain $F(u,t)$ by performing the substitution (2.3).

The following direct argument also gives $F(u,t)$: a walk on the ladder is obtained from a walk on a pole (i.e. on $\mathbf{Z}$) by inserting before and after every step on a pole a (possibly empty) sequence of steps from one pole to the other. This process is expressed by performing on $F_{\mathbf{Z}}$ the substitution

$$t \mapsto t + t^2 + t^3u + t^4u^2 + \cdots = t + \frac{t^2}{1-tu}\,,$$

corresponding to replacing a step on a pole by itself, or itself followed by a step to the other pole, or itself, a step to the other pole and a step back, etc. But if the path had a bump at the place the substitution was performed, this bump would disappear when a step is added from one pole to the other. In formulas,

$$tu \mapsto tu + t^2 + t^3u + t^4u^2 + \cdots = tu + \frac{t^2}{1-tu}\,.$$

Finally we must add at the beginning of the path a sequence of steps from one pole to the other. Therefore we obtain

$$F(u,t) = \text{even}\Big\{\Big(1 + \frac{t}{1-tu}\Big)F_{\mathbf{Z}}\Big(\frac{tu + t^2/(1-tu)}{t + t^2/(1-tu)}, t + \frac{t^2}{1-tu}\Big)\Big\}\,.$$

As another example, consider the group $\mathbf{Z}$ generated by the non-free set $\{\pm 1, \pm 2\}$. Geometrically, it can be seen as the set of points $(2i, 0)$ and $(2i+1, \sqrt{3})$ for all $i \in \mathbf{Z}$, with edges between all points at Euclidean distance 2 apart; but we will not make use of this description. The circuit series of $\mathbf{Z}$ with this enlarged generating set will be an algebraic function of degree 4 over the rationals.

Define first the following series:

$f(t)$ counts the walks from 0 to 0 in $\mathbf{N}$;

$g(t)$ counts the walks from 0 to 1 in $\mathbf{N}$;

$h(t)$ counts the walks from 1 to 1 in $\mathbf{N}$.

Denote the generators of $\mathbf{Z}$ by $1 = \uparrow$, $2 = \uparrow\uparrow$, $-1 = \downarrow$ and $-2 = \downarrow\downarrow$. The series then satisfy the following equations, where the generators' symbol is written instead of 't' to make the formulas self-explanatory:

$$f = 1 + (\uparrow f \downarrow + \uparrow g \downarrow\downarrow + \uparrow\uparrow g \downarrow + \uparrow\uparrow h \downarrow\downarrow)f\,,$$
$$g = f \uparrow f + f \uparrow\uparrow g\,,$$
$$h = f + f \downarrow g + g \downarrow\downarrow g\,,$$

giving a solution f that is algebraic of degree 4 over $\mathbf{Z}(t)$.

Then define the following series:

G counts the walks from 0 to 0 in $\mathbf{Z}$;

e counts the walks from 0 to 1 in $\mathbf{Z}$.

They satisfy the equations

$$G = 1 + 2(\uparrow f \downarrow G + \upuparrows g \downarrow G + \uparrow f \downdownarrows e + \upuparrows g \downdownarrows e + \uparrow g \downdownarrows G + \upuparrows h \downdownarrows G),$$
$$e = G \uparrow f + G \upuparrows f + G \upuparrows g$$

giving the solution

$$G = \frac{4 + 3t - 6t^2 - 10t(1+2t)\delta + 2t^2(3+8t)\delta^2 - 6t^4(1+t)\delta^3}{4 - 7t - 36t^2}$$

where δ is a root of the equation

$$1 - (2t+1)\delta + t(2+3t)\delta^2 - t^2(1+2t)\delta^3 + t^4\delta^4 = 0 .$$

8. Cogrowth of non-free presentations

We perform here a computation extending the results of Section 3.1. The general setting, expressed in the language of group theory, is the following: let Π be a group generated by a finite set S and let $\Xi < \Pi$ be any subgroup. We consider the following generating series:

$$F(t) = \sum_{\gamma \in \Xi < \Pi} t^{|\gamma|},$$
$$G(t) = \sum_{\substack{\text{words } w \text{ in } S \\ \text{defining an element in } \Xi}} t^{|w|},$$

where $|\gamma|$ is the minimal length of γ in the generators S, and $|w|$ is the usual length of the word w. Is there some relation between these series? In case Π is quasi-free on S, the relation between F and G is given by Corollary 2.6. We consider two other examples: Π quasi-free but on a set smaller than S, and $\Pi = \mathbf{PSL}_2(\mathbf{Z})$.