

## 2. A COUPLE OF FACTS FROM COHOMOLOGY

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COROLLARY 2. *Let  $\Sigma$  be a finite subset of places of  $\mathbf{Q}$ . Assume that*

- (a) *For all places  $v \notin \Sigma$  the function  $f \in K$  is a norm from  $\mathbf{Q}_v L$  to  $\mathbf{Q}_v(t)$ .*
- (b) *For all  $v \in \Sigma$  there exist  $a_v, b_{i,v} \in \mathbf{Q}_v$  with*

$$f(a_v) = N(a_v, b_{1,v}, \dots, b_{d,v}) \in \mathbf{Q}_v^* .$$

*Then  $f$  is a norm from  $L$ .*

(In §5 we shall see that (a) is not sufficient in itself.) Colliot-Thélène has shown me a different proof of this corollary using the above-mentioned Faddeev exact sequence, actually removing the regularity assumption. The result reminds one of the work by Pourchet (see [Raj, Lemma 17.4]) and by Colliot-Thélène, Coray, Sansuc [CThCS, Prop. 1.3]. (For instance the last paper contains the proof that a *multiplicative* quadratic form over  $k(t)$  represents  $f$  over  $k(t)$  if and only if it represents  $f$  over  $k_v(t)$  for all places  $v$  of  $k$ .)

The paper is organized as follows. In §2 we shall recall a few basics from cohomology. In §3 we shall prove the theorem and its corollaries. In §4 we shall discuss a simple counterexample to an analogous result when  $\text{Gal}(L/K)$  is a four-group (similarly to the number-field case). In §5 we shall discuss how the assumptions for Corollary 2 are equivalent for large  $p$  both to the solvability of congruences  $f \equiv N(g) \pmod{p}$  and to the existence of solutions over the completion of  $\mathbf{Q}_p L$  under the Gauss norm. Incidentally, we shall prove that if a representation of  $f$  by  $N$  exists at all with the  $x_i \in k(t)$ , then some representation will have the  $x_i$ 's of degree bounded explicitly only in terms of  $\deg f$  and genus and degree of  $kL/k(t)$ . This seems to have some interest in itself. These observations lead also to the construction of varieties satisfying the usual local-global principle. Finally, in §6 we shall discuss how to find effectively a possible representation of  $f$  by  $N$ .

## 2. A COUPLE OF FACTS FROM COHOMOLOGY

Let  $G$  be a finite group acting on an abelian group  $M$ . For a function  $\xi: G \rightarrow M$ ,  $\sigma \mapsto \xi_\sigma$  we denote (the usual coboundary operator)

$$\partial(\xi_\sigma) = \partial(\xi): G^2 \rightarrow M, \quad (\sigma, \tau) \mapsto \xi_\sigma + \sigma(\xi_\tau) - \xi_{\sigma\tau} .$$

With this notation (but writing  $M$  multiplicatively) we now recall Hilbert's Theorem 90:

Let  $k_1/k$  be a finite Galois extension with group  $G$  and let  $\xi: G \rightarrow k_1^*$  be a function satisfying  $\partial(\xi) = 1$ . Then there exists  $\alpha \in k_1^*$  such that  $\xi_\sigma = \alpha/\sigma(\alpha)$  for all  $\sigma \in G$ .

The usual proof (see e.g. [CF, Prop. 3, p.124]) is simple and runs as follows: For  $x \in k_1$  form the sum  $\alpha = \sum_{\sigma \in G} \xi_\sigma \sigma(x)$ . By a well-known elementary result of Artin, we may choose  $x \in k_1$  such that  $\alpha \neq 0$ . A quick computation using the assumption on  $\xi$  then shows that  $\alpha$  has the stated property.

An easy corollary (the original Hilbert's 90) is that, if  $G$  is cyclic generated by  $g$ , then every element  $a \in k_1^*$  such that  $N_k^{k_1}(a) = 1$  is of the form  $b/g(b)$  for some  $b \in k_1^*$ . To derive this conclusion it suffices to apply the above statement to the function on  $G$  defined by  $\xi_{g^m} = \prod_{i=0}^{m-1} g^i(a)$  (which is well defined).

In §6 on effectiveness we shall need a simple result on *permutation modules* for the action of a finite group  $G$ . Such a module is simply a free abelian group on which  $G$  acts, which moreover has a  $\mathbf{Z}$ -basis permuted by  $G$ . We have:

Let  $M$  be a permutation module and let  $\xi: G \rightarrow M$  satisfy  $\partial(\xi) = 0$ . Then there exists  $m \in M$  such that  $\xi_\sigma = m - \sigma(m)$  for all  $\sigma \in G$ .

We give a short argument for completeness. We may write  $M$  as a direct sum of permutation modules, each of which has a  $\mathbf{Z}$ -basis which is a  $G$ -orbit. It suffices to prove the claim for each direct factor. Write the mentioned basis as  $\{g(b)\}$  for a certain  $b \in M$  and  $g$  running through a set of representatives for  $G/H$ ,  $H$  being the stabilizer of  $b$ .

We sum the equations  $\xi_{\sigma\tau} = \xi_\sigma + \sigma(\xi_\tau)$  over  $\tau \in G$ . Letting  $n$  be the order of  $G$  and putting  $\mu := \sum_{g \in G} \xi_g \in M$ , we get

$$n\xi_\sigma = \mu - \sigma(\mu).$$

Write  $\mu = \sum_{g \in G/H} a_g g(b)$  for suitable  $a_g \in \mathbf{Z}$ . The displayed equation implies  $\mu \equiv \sigma(\mu) \pmod{nM}$  for every  $\sigma \in G$ . This immediately gives the existence of  $a \in \mathbf{Z}$  such that  $a_g \equiv a \pmod{n}$  for all  $g \in G/H$ , so we write  $a_g = a + nq_g$  where  $q_g \in \mathbf{Z}$ . Let  $m := \sum_{G/H} q_g g(b) \in M$ . Then  $nm = \mu - a \sum_{G/H} g(b)$ , where the last term is invariant by  $G$ . Hence  $n\xi_\sigma = n(m - \sigma(m))$ , whence  $\xi_\sigma = m - \sigma(m)$ , as required.