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Autor: Mattner, Lutz

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2.2 A FUBINI COUNTEREXAMPLE

In this section, we give an example of (2). Let $\mathcal{A}$ be as in (5) and define $\mu|_{\mathcal{A}}$ by

$$\mu(A) := \begin{cases} 0 & \text{if } A \text{ is meager}, \\ 1 & \text{if } A \text{ is comeager}. \end{cases} \tag{11}$$

This is possible, since no set $A \subset \mathbf{R}$ is simultaneously meager and comeager, for otherwise $\varnothing = A \cap A^c$ would be comeager, in contradiction to Baire's theorem. It is easy to check that μ is a probability measure on $(\mathbf{R}, \mathcal{A})$. Let again $\nu := \lambda :=$ Lebesgue measure on $\mathcal{B} := \mathcal{B}(\mathbf{R})$, and choose $A \in \mathcal{A}$ meager with $\lambda(A^c) = 0$. Then $1_A(\cdot + y)$ is $\mathcal{A}$-measurable with

$$\int_{\mathbf{R}} 1_A(x+y)\, d\mu(x) = \mu(A-y) = 0 \qquad (y \in \mathbf{R}).$$

On the other hand, we have

$$\int_{\mathbf{R}} 1_A(x+y)\, d\nu(y) = \lambda(A-x) = \infty \qquad (x \in \mathbf{R}).$$

Hence (2) is obviously true in this case.

3. MEASURABILITY

Here is a positive result, having a certain measurability property of F from (1) among its conclusions. An application of this occurs in Mattner (1999).

3.1. THEOREM. *Let $(\mathcal{X}, \mathcal{A}, \mu)$ and $(\mathcal{Y}, \mathcal{B}, \nu)$ be σ-finite measure spaces, let $f\colon \mathcal{X} \times \mathcal{Y} \to [0, \infty]$ be a function measurable with respect to the product σ-algebra $\mathcal{A} \otimes \mathcal{B}$, and put*

$$\begin{aligned} \mathcal{A}_0 &:= \sigma(\{f(\cdot, y) \,:\, y \in \mathcal{Y}\}), \\ \mathcal{B}_0 &:= \sigma(\{f(x, \cdot) \,:\, x \in \mathcal{X}\}), \\ \overline{\mathcal{A}}_0 &:= \{A \in \mathcal{A} \,:\, \exists A_0 \in \mathcal{A}_0 \text{ with } A = A_0 \quad [\mu]\}, \\ \overline{\mathcal{B}}_0 &:= \{B \in \mathcal{B} \,:\, \exists B_0 \in \mathcal{A}_0 \text{ with } B = B_0 \quad [\nu]\}, \\ \overline{\mathcal{A}_0 \otimes \mathcal{B}_0} &:= \{C \in \mathcal{A} \otimes \mathcal{B} \,:\, \exists C_0 \in \mathcal{A}_0 \otimes \mathcal{B}_0 \text{ with } C = C_0 \quad [\mu \otimes \nu]\}. \end{aligned}$$

Then f is $\overline{\mathcal{A}_0 \otimes \mathcal{B}_0}$-measurable, $\int_{\mathcal{Y}} f(\cdot, y)\, d\nu(y)$ is $\overline{\mathcal{A}}_0$-measurable, and $\int_{\mathcal{X}} f(x, \cdot)\, d\mu(x)$ is $\overline{\mathcal{B}}_0$-measurable.

Here and in what follows, we write $A = A_0 \quad [\mu]$ for $\mu(A \triangle A_0) = 0$. Below we also use the corresponding notation $f = g \quad [\mu]$ for functions, meaning $\mu(\{x : f(x) \neq g(x)\}) = 0$.

3.2 Remarks

Let us retain the notation and assumptions of Theorem 3.1.

a) The parameter integral $\int_{\mathcal{Y}} f(\cdot, y)\, d\nu(y)$ need not be $\mathcal{A}_0$-measurable and f need not be $\mathcal{A}_0 \otimes \mathcal{B}_0$-measurable, as the example in 2.1 shows.

b) The function f need not be $\overline{\mathcal{A}}_0 \otimes \overline{\mathcal{B}}_0$-measurable. As an example proving this remark, we may take $(\mathcal{X}, \mathcal{A}, \mu) := (\mathcal{Y}, \mathcal{B}, \nu) := ([0,1], \mathcal{B}([0,1]), \lambda^1)$, $D := \{(x,x) : x \in [0,1]\}$, and $f := 1_D$. [We now write λ^d for d-dimensional Lebesgue measure.] Then

$$\begin{aligned} \mathcal{A}_0 &= \mathcal{B}_0 = \{A \in \mathcal{B}([0,1]) : A \text{ countable or cocountable}\}\ , \\ \overline{\mathcal{A}}_0 &= \overline{\mathcal{B}}_0 = \left\{A \in \mathcal{B}([0,1]) : \lambda^1(A) \in \{0,1\}\right\}\ , \end{aligned}$$

and we claim that f is not $\overline{\mathcal{A}}_0 \otimes \overline{\mathcal{B}}_0$-measurable. To prove this, put

$$\mathcal{C} := \left\{C \in \mathcal{B}([0,1]^2) : \left(\lambda^2(C), \int_0^1 1_C(x,x)\, d\lambda^1(x)\right) \in \{(0,0),(1,1)\}\right\}.$$

Then $\mathcal{C}$ is a σ-algebra containing $\{A \times B : A \in \overline{\mathcal{A}}_0,\ B \in \overline{\mathcal{B}}_0\}$, and hence satisfies $\overline{\mathcal{A}}_0 \otimes \overline{\mathcal{B}}_0 \subset \mathcal{C}$. But $D \notin \mathcal{C}$, so that $D \notin \overline{\mathcal{A}}_0 \otimes \overline{\mathcal{B}}_0$.

c) Let us write more explicitly $\overline{\mathcal{A}}_0(\mu)$ in place of $\overline{\mathcal{A}}_0$. From Theorem 3.1, we may deduce the measurability of $F := \int f(\cdot, y)\, d\nu(y)$ with respect to $\bigcap_\mu \overline{\mathcal{A}}_0(\mu)$, the intersection being over all $\mathcal{A}$ and μ as in the theorem. This, however, must not be confused with the more restrictive property of universal $\mathcal{A}_0$-measurability of F [see Cohn (1980), pages 280–283, for the definition and for illuminating facts]. Indeed, our measures μ are supposed to be defined on some $\mathcal{A}$ rendering f $\mathcal{A} \otimes \mathcal{B}$-measurable, and not merely on $\mathcal{A}_0$ or its μ-completion. For example, in the situation of 2.1, one can use the measure μ from (11) to deduce that the σ-algebra of all universally $\mathcal{A}_0$-measurable sets is contained in $\widetilde{\mathcal{A}}_0 := \{A \subset \mathbf{R} : A \text{ meager or comeager}\}$. Since $\widetilde{\mathcal{A}}_0$ differs from $\mathcal{A}_0$ only by non-Borel sets, we see that F from (3), (7), (8) is not universally $\mathcal{A}_0$-measurable. By the way, the known fact that μ from (11) can not be extended to a measure on $\mathcal{B}(\mathbf{R})$ [see Oxtoby (1980), page 86] follows from our present considerations, since otherwise we would have $\overline{\mathcal{A}}_0(\mu) = \widetilde{\mathcal{A}}_0 \cap \mathcal{B}(\mathbf{R}) = \mathcal{A}_0$, and Theorem 3.1 would yield $\mathcal{A}_0$-measurability of F.

3.3 PROOF OF THEOREM 3.1

Obvious arguments show that we may assume in addition that

(12) μ, ν are finite and f is bounded.

The proof of the theorem splits into two parts as follows.

CLAIM 1. *Under the assumptions of the theorem and* (12)*,*

$$F := \int f(\cdot, y)\, d\nu(y) \tag{13}$$

is $\overline{\mathcal{A}}_0$*-measurable.*

Proof. Let us first recall the "mean value theorem" for vector valued integration: Let E be a topological vector space, $(\Omega, \mathcal{A}, \mu)$ be a measure space, and $g\colon \Omega \to E$ be a function. Then an $x \in E$ is called the *weak* (or *Pettis*) integral of g, and we write $\int g\, d\mu := x$, if

(i) the dual space E' of E separates points on E,

(ii) the scalar function $\langle y, g(\cdot)\rangle$ belongs to $\mathcal{L}^1(\Omega, \mathcal{A}, \mu)$ for every $y \in E'$, and

(iii) $\int \langle y, g(\omega)\rangle\, d\mu(\omega) = \langle y, x\rangle$ for every $y \in E'$.

[This is the definition adopted by Edwards (1965), p. 566, and by Rudin (1991), p. 77.] If now E is in particular locally convex Hausdorff and μ is bounded, then the weak integral, if it exists, necessarily satisfies

$$\int g\, d\mu \in \mu(\Omega) \cdot \overline{\operatorname{conv}}\, g(\Omega)\,, \tag{14}$$

with $\overline{\operatorname{conv}}$ indicating convex closure. This "mean value theorem" is surely well known. It follows easily from the Hahn-Banach theorem: Apply Theorem 3.4 (b) of Rudin (1991) to $A := \left\{\int g\, d\mu\right\}$ and $B := \mu(\Omega) \cdot \overline{\operatorname{conv}}\, g(\Omega)$.

We now start with the proof proper. The functions $f(\cdot, y)\colon \mathcal{X} \to \mathbf{R}$, as well as F from (13), are $\mathcal{A}$-measurable [by $\mathcal{A}\otimes\mathcal{B}$-measurability of f and by Fubini] and bounded, and hence belong to $\mathcal{L}^1(\mathcal{X}, \mathcal{A}, \mu)$. Let $[f(\cdot, y)], [F] \in L^1(\mathcal{X}, \mathcal{A}, \mu)$ denote their corresponding equivalence classes. We claim that

$$[F] = \int_{\mathcal{Y}} [f(\cdot, y)]\, d\nu(y)\,, \tag{15}$$

in the weak sense recalled above, applied to the Banach space $E = L^1(\mathcal{X}, \mathcal{A}, \mu)$ with dual space $L^\infty(\mathcal{X}, \mathcal{A}, \mu)$. To prove this, let $h \in [h] \in L^\infty(\mathcal{X}, \mathcal{A}, \mu)$. An obvious Fubini calculation, using the definition of F and the $\mathcal{A}\otimes\mathcal{B}$-measurability of f, yields

$$\langle [h], [F] \rangle = \int_{\mathcal{X}} h(x)F(x)\, d\mu(x) = \int_{\mathcal{Y}} \langle [h], [f(\cdot, y)] \rangle\, d\nu(y),$$

which confirms (15). [Actually, (15) is even true with the right hand side read as a Bochner integral, but we do not need this fact here.] We now use that each $f(\cdot, y)$ is $\mathcal{A}_0$-measurable, where of course $\mathcal{A}_0 \subset \mathcal{A}$. This implies that the function $y \mapsto [f(\cdot, y)]$ takes its values in

$$S := \{\Phi \in L^1(\mathcal{X}, \mathcal{A}, \mu) \, : \, \exists\, \mathcal{A}_0\text{-measurable } \varphi \in \Phi\},$$

which is easily seen to be a closed subspace of $L^1(\mathcal{X}, \mathcal{A}, \mu)$. The mean value theorem (14) now yields $[F] \in S$, which is the desired conclusion. □

CLAIM 2. *Under the assumptions of the theorem and* (12), *and assuming the truth of Claim 1, f is $\overline{\mathcal{A}_0 \otimes \mathcal{B}_0}$-measurable.*

Proof. We consider the restrictions

$$\overline{\mu}_0 := \mu|_{\overline{\mathcal{A}}_0}, \qquad \overline{\nu}_0 = \nu|_{\overline{\mathcal{A}}_0},$$

and define a function $\tau \colon \overline{\mathcal{A}}_0 \otimes \overline{\mathcal{B}}_0 \to [0, \infty]$ by

$$\tau(C) := \int_{\mathcal{X}} \int_{\mathcal{Y}} f(x, y) 1_C(x, y)\, d\overline{\nu}_0(y)\, d\overline{\mu}_0(x) \qquad (C \in \overline{\mathcal{A}}_0 \otimes \overline{\mathcal{B}}_0), \tag{16}$$

and we emphasize that the right hand side has to be read as an iterated integral. In order to show its existence, we have to check that the function $x \mapsto \int_{\mathcal{Y}} f(x, y) 1_C(x, y)\, d\overline{\nu}_0(y)$ is $\overline{\mathcal{A}}_0$-measurable. For the special case $C = A \times B$ with $A \in \overline{\mathcal{A}}_0$ and $B \in \overline{\mathcal{B}}_0$, this follows from Claim 1, applied to $\overline{\mathcal{A}}_0$ in place of $\mathcal{A}_0$ and $f(x, y) 1_B(y)$ in place of $f(x, y)$, and using $\overline{\overline{\mathcal{A}}}_0 = \overline{\mathcal{A}}_0$. The general case follows as usual via Sierpiński's lemma [Satz I.6.8 in Elstrodt (1996)]. Thus τ is well-defined. It is easily checked that τ is a measure, and that every set of $\overline{\mu}_0 \otimes \overline{\nu}_0$-measure zero is of τ-measure zero as well. Hence the Lebesgue-Radon-Nikodym theorem yields the existence of an $\overline{\mathcal{A}}_0 \otimes \overline{\mathcal{B}}_0$-measurable function $\widetilde{f} \colon \mathcal{X} \times \mathcal{Y} \to [0, \infty]$ such that

$$\tau(C) = \int_C \widetilde{f}\, d\overline{\mu}_0 \otimes \overline{\nu}_0 \qquad (C \in \overline{\mathcal{A}}_0 \otimes \overline{\mathcal{B}}_0).$$

By (16) and Fubini, this implies in particular

$$\int_{A_0} \int_{B_0} f(x, y)\, d\overline{\nu}_0(y)\, d\overline{\mu}_0(x) = \int_{A_0} \int_{B_0} \widetilde{f}(x, y) d\overline{\nu}_0(y)\, d\overline{\mu}_0(x) \tag{17}$$

($A_0 \in \overline{\mathcal{A}}_0$, $B_0 \in \overline{\mathcal{B}}_0$). Since, using (12), both sides in (17) are always finite, we may conclude for every $B_0 \in \overline{\mathcal{B}}_0$:

$$\int_{B_0} f(\cdot, y)\, d\overline{\nu}_0(y) = \int_{B_0} \widetilde{f}(\cdot, y)\, d\overline{\nu}_0(y) \qquad [\overline{\mu}_0]\,.$$

Trivially, this remains true if $[\overline{\mu}_0]$ is replaced by $[\mu]$, and an integration yields

$$\int\limits_{A}\int\limits_{B_0} f(x, y)\, d\overline{\nu}_0(y)\, d\mu(x) = \int\limits_{A}\int\limits_{B_0} \widetilde{f}(x, y) d\overline{\nu}_0(y)\, d\mu(x) \tag{18}$$

$(A \in \mathcal{A},\ B_0 \in \overline{\mathcal{B}}_0)$. We now want to interchange the order of integrations. Since $\widetilde{f}$ is trivially $\mathcal{A} \otimes \overline{\mathcal{B}}_0$-measurable, we may obviously do this on the right hand side of (18). To do the same on the left hand side, we rewrite it successively as

$$\int_A\int_{B_0} f(x, y)\, d\nu(y)\, d\mu(x) = \int_{B_0}\int_A f(x, y)\, d\mu(x)\, d\nu(y) = \int_{B_0}\int_A f(x, y)\, d\mu(x)\, d\overline{\nu}_0(y)\,,$$

where the last equality follows from a second application of Claim 1, with the role of the variables interchanged. Thus (18) yields

$$\int\limits_{B_0}\int\limits_{A} f(x, y)\, d\mu(x)\, d\overline{\nu}_0(y) = \int\limits_{B_0}\int\limits_{A} \widetilde{f}(x, y) d\mu(x)\, d\overline{\nu}_0(y) \tag{19}$$

$(A \in \mathcal{A},\ B_0 \in \overline{\mathcal{B}}_0)$. Now the argument leading from (17) to (18) can be repeated to lead from (19) to a corresponding statement with B in place of B_0, ν in place of $\overline{\nu}_0$, and $\mathcal{B}$ in place of $\overline{\mathcal{B}}_0$, which is equivalent to

$$\int_{A\times B} f\, d\mu \otimes \nu = \int_{A\times B} \widetilde{f}\, d\mu \otimes \nu \qquad (A \in \mathcal{A},\, B \in \mathcal{B})\,.$$

This shows that $f = \widetilde{f}\quad [\mu \otimes \nu]$, which yields the desired conclusion. $\square$

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