

2. Relative effective Cartier divisors

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1. A THEOREM OF GROTHENDIECK

The following theorem is a special case of Grothendieck's theorems, and the proof can be found in [Mu] §5, [H] §3.12, or [EGA] III, §7.7.5, 7.9.4.

THEOREM 1.1. *Let $q: V \rightarrow T$ be a proper flat morphism of noetherian schemes and let \mathcal{L} be an invertible sheaf on V . For each $t \in T$ denote the fiber $V \otimes_T \text{spec}(k(t))$ of q at t by V_t , where $k(t)$ is the residue field of T at t . Denote the inverse image of \mathcal{L} on V_t by \mathcal{L}_t .*

- (a) *The function $t \mapsto \chi(\mathcal{L}_t) = \sum_i (-1)^i \dim_{k(t)} H^i(V_t, \mathcal{L}_t)$ is locally constant on T .*
- (b) *For each i , the function $t \mapsto \dim_{k(t)} H^i(V_t, \mathcal{L}_t)$ on T is upper semicontinuous.*
- (c) *If T is reduced and connected and if $t \mapsto \dim_{k(t)} H^i(V_t, \mathcal{L}_t)$ is a constant function on T , then $R^i q_* \mathcal{L}$ is a locally free sheaf on T and the map $R^i q_* \mathcal{L} \otimes_{\mathcal{O}_T} k(t) \rightarrow H^i(V_t, \mathcal{L}_t)$ is an isomorphism.*
- (d) *If $H^1(V_t, \mathcal{L}_t) = 0$ for all $t \in T$, then $R^1 q_* \mathcal{L} = 0$ and $q_* \mathcal{L}$ is a locally free sheaf. Moreover the formation of $q_* \mathcal{L}$ commutes with any base change.*

2. RELATIVE EFFECTIVE CARTIER DIVISORS

Let $q: X \rightarrow T$ be a morphism of noetherian schemes. A *relative effective Cartier divisor* on X/T is an effective Cartier divisor on X that is flat over T when regarded as a closed subscheme of X . When $T = \text{spec}(R)$ is affine, a closed subscheme D of X is a relative effective Cartier divisor if and only if there exists an open affine covering $U_i = \text{spec}(R_i)$ of X and $g_i \in R_i$ such that

- (a) $D \cap U_i = \text{spec}(R_i/(g_i))$;
- (b) g_i is not a zero divisor;
- (c) $R_i/(g_i)$ is flat over R .

REMARK 2.1. Let D be an effective Cartier divisor on X/T , let $\mathcal{I}(D)$ be the sheaf of ideals defining D , and let $\mathcal{L}(D)$ be the invertible sheaf corresponding to D . We have $\mathcal{L}(D) = \mathcal{I}(D)^{-1}$. The inclusion $\mathcal{I}(D) \subset \mathcal{O}_X$ induces $\mathcal{O}_X \subset \mathcal{I}(D)^{-1} = \mathcal{L}(D)$, hence a section s_D of $\mathcal{L}(D)$.

The map $D \mapsto (\mathcal{L}(D), s_D)$ defines a one-to-one correspondence between the set of relative effective Cartier divisors on X/T and the isomorphism classes of pairs (\mathcal{L}, s) , where \mathcal{L} is an invertible sheaf on X and s is a global section of \mathcal{L} such that the map $s: \mathcal{O}_X \rightarrow \mathcal{L}$ induced by the section s is injective and $\mathcal{L}/s\mathcal{O}_X$ is \mathcal{O}_T -flat.

The proof of the following lemma is straightforward and is left to the reader:

LEMMA 2.2.

(a) If D_1 and D_2 are relative effective Cartier divisors on X/T , then so is $D_1 + D_2$.

(b) Let D_1 and D_2 be two relative effective Cartier divisors on X/T and let $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$ be their ideal sheaves. If $\mathcal{I}(D_1) \subset \mathcal{I}(D_2)$, then $D_1 - D_2$ is also a relative effective Cartier divisor on X/T .

(c) Let $T' \rightarrow T$ be a base extension and let $X' = X \times_T T'$. If D is a relative effective Cartier divisor on X/T , then its pull-back to a closed subscheme D' of X' is a relative effective Cartier divisor on X'/T' .

LEMMA 2.3. Assume $q: X \rightarrow T$ is flat. Let \mathcal{I} be a coherent sheaf of ideals of \mathcal{O}_X and let D be the closed subscheme of X defined by \mathcal{I} . If for every point $x \in D$, the ideal \mathcal{I}_x of $\mathcal{O}_{X,x}$ is generated by one element g_x whose image in $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{T,q(x)}} k(q(x))$ is not a zero divisor, then D is a relative effective Cartier divisor.

Proof. It suffices to show that g_x is not a zero divisor in $\mathcal{O}_{X,x}$ and that $\mathcal{O}_{X,x}/(g_x)$ is flat over $\mathcal{O}_{T,q(x)}$. This follows from [EGA] §0.10.2.4 by taking $A = \mathcal{O}_{T,q(x)}$, $B = \mathcal{O}_{X,x}$, $M = N = \mathcal{O}_{X,x}$, and $u: M \rightarrow N$ to be the homomorphism $g_x: \mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X,x}$ defined by the multiplication by g_x .

3. THE CONSTRUCTION OF A BIRATIONAL GROUP

Let X be a nonsingular irreducible projective curve over an algebraically closed field k . A *modulus* \mathfrak{m} supported on a finite subset S of X is a divisor of the form $\mathfrak{m} = \sum_{P \in S} n_P P$ with each $n_P > 0$. For any rational function f on X , we write $f \equiv 0 \pmod{\mathfrak{m}}$ if $v_P(f) \geq n_P$ for every $P \in S$, where v_P is the valuation defined by P . Two divisors D_1 and D_2 on X prime to S are called *m-equivalent* if there exists a rational function f satisfying $f - 1 \equiv 0 \pmod{\mathfrak{m}}$ such that $D_1 - D_2 = (f)$. If this holds, we write $D_1 \sim_{\mathfrak{m}} D_2$. Define a ringed