2. Relative effective Cartier divisors

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1. A THEOREM OF GROTHENDIECK

The following theorem is a special case of Grothendieck's theorems, and the proof can be found in [Mu] §5, [H] §3.12, or [EGA] III, §7.7.5, 7.9.4.

THEOREM 1.1. *Let* $q\colon V \to T$ *be a proper flat morphism of noetherian schemes and let* $\mathcal{L}$ *be an invertible sheaf on* V*. For each* $t \in T$ *denote the fiber* $V \otimes_T \operatorname{spec}(k(t))$ *of* q *at* t *by* V_t*, where* $k(t)$ *is the residue field of* T *at* t*. Denote the inverse image of* $\mathcal{L}$ *on* V_t *by* $\mathcal{L}_t$*.*

(a) *The function* $t \mapsto \chi(\mathcal{L}_t) = \sum_i (-1)^i \dim_{k(t)} H^i(V_t, \mathcal{L}_t)$ *is locally constant on* T*.*

(b) *For each* i*, the function* $t \mapsto \dim_{k(t)} H^i(V_t, \mathcal{L}_t)$ *on* T *is upper semicontinuous.*

(c) *If* T *is reduced and connected and if* $t \mapsto \dim_{k(t)} H^i(V_t, \mathcal{L}_t)$ *is a constant function on* T*, then* $R^i q_* \mathcal{L}$ *is a locally free sheaf on* T *and the map* $R^i q_* \mathcal{L} \otimes_{\mathcal{O}_T} k(t) \to H^i(V_t, \mathcal{L}_t)$ *is an isomorphism.*

(d) *If* $H^1(V_t, \mathcal{L}_t) = 0$ *for all* $t \in T$*, then* $R^1 q_* \mathcal{L} = 0$ *and* $q_* \mathcal{L}$ *is a locally free sheaf. Moreover the formation of* $q_* \mathcal{L}$ *commutes with any base change.*

2. RELATIVE EFFECTIVE CARTIER DIVISORS

Let $q\colon X \to T$ be a morphism of noetherian schemes. A *relative effective Cartier divisor* on X/T is an effective Cartier divisor on X that is flat over T when regarded as a closed subscheme of X. When $T = \operatorname{spec}(R)$ is affine, a closed subscheme D of X is a relative effective Cartier divisor if and only if there exists an open affine covering $U_i = \operatorname{spec}(R_i)$ of X and $g_i \in R_i$ such that

(a) $D \cap U_i = \operatorname{spec}\big(R_i/(g_i)\big)$;
(b) g_i is not a zero divisor;
(c) $R_i/(g_i)$ is flat over R.

REMARK 2.1. Let D be an effective Cartier divisor on X/T, let $\mathcal{I}(D)$ be the sheaf of ideals defining D, and let $\mathcal{L}(D)$ be the invertible sheaf corresponding to D. We have $\mathcal{L}(D) = \mathcal{I}(D)^{-1}$. The inclusion $\mathcal{I}(D) \subset \mathcal{O}_X$ induces $\mathcal{O}_X \subset \mathcal{I}(D)^{-1} = \mathcal{L}(D)$, hence a section s_D of $\mathcal{L}(D)$.

The map $D \mapsto (\mathcal{L}(D), s_D)$ defines a one-to-one correspondence between the set of relative effective Cartier divisors on X/T and the isomorphism classes of pairs $(\mathcal{L}, s)$, where $\mathcal{L}$ is an invertible sheaf on X and s is a global section of $\mathcal{L}$ such that the map $s\colon \mathcal{O}_X \to \mathcal{L}$ induced by the section s is injective and $\mathcal{L}/s\mathcal{O}_X$ is $\mathcal{O}_T$-flat.

The proof of the following lemma is straightforward and is left to the reader:

LEMMA 2.2.

(a) *If D_1 and D_2 are relative effective Cartier divisors on X/T, then so is $D_1 + D_2$.*

(b) *Let D_1 and D_2 be two relative effective Cartier divisors on X/T and let $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$ be their ideal sheaves. If $\mathcal{I}(D_1) \subset \mathcal{I}(D_2)$, then $D_1 - D_2$ is also a relative effective Cartier divisor on X/T.*

(c) *Let $T' \to T$ be a base extension and let $X' = X \times_T T'$. If D is a relative effective Cartier divisor on X/T, then its pull-back to a closed subscheme D' of X' is a relative effective Cartier divisor on X'/T'.*

LEMMA 2.3. *Assume $q\colon X \to T$ is flat. Let $\mathcal{I}$ be a coherent sheaf of ideals of $\mathcal{O}_X$ and let D be the closed subscheme of X defined by $\mathcal{I}$. If for every point $x \in D$, the ideal $\mathcal{I}_x$ of $\mathcal{O}_{X,x}$ is generated by one element g_x whose image in $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{T,q(x)}} k(q(x))$ is not a zero divisor, then D is a relative effective Cartier divisor.*

Proof. It suffices to show that g_x is not a zero divisor in $\mathcal{O}_{X,x}$ and that $\mathcal{O}_{X,x}/(g_x)$ is flat over $\mathcal{O}_{T,q(x)}$. This follows from [EGA] §0.10.2.4 by taking $A = \mathcal{O}_{T,q(x)}$, $B = \mathcal{O}_{X,x}$, $M = N = \mathcal{O}_{X,x}$, and $u\colon M \to N$ to be the homomorphism $g_x\colon \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$ defined by the multiplication by g_x.

3. THE CONSTRUCTION OF A BIRATIONAL GROUP

Let X be a nonsingular irreducible projective curve over an algebraically closed field k. A *modulus* $\mathfrak{m}$ supported on a finite subset S of X is a divisor of the form $\mathfrak{m} = \sum_{P \in S} n_P P$ with each $n_P > 0$. For any rational function f on X, we write $f \equiv 0 \mod \mathfrak{m}$ if $v_P(f) \geq n_P$ for every $P \in S$, where v_P is the valuation defined by P. Two divisors D_1 and D_2 on X prime to S are called $\mathfrak{m}$-*equivalent* if there exists a rational function f satisfying $f - 1 \equiv 0 \mod \mathfrak{m}$ such that $D_1 - D_2 = (f)$. If this holds, we write $D_1 \sim_{\mathfrak{m}} D_2$. Define a ringed