8. The 3d distance theorem

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **44 (1998)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **21.09.2024**

7.2 APPLICATION TO BINARY CODINGS

A more natural coding of the rotation R would have been with respect to the partition $[0,\beta[, [\beta,1[$. The points $\{0\},\{\beta\},\{\alpha\},\{\beta+\alpha\},\ldots,\{n\alpha\}$, $\{\beta+n\alpha\}$ are the endpoints of the sets $I(w_1,\ldots,w_n)$, following the notation of Section 2. But these sets might not be connected. Thus the frequencies of factors of length n are the sums of the lengths of the connected components of the sets $I(w_1,\ldots,w_n)$. Despite this disadvantage, this coding allows us to deduce the following result from Lemma 3.

THEOREM 19. *Let u be a coding of an irrational rotation with respect to the partition into two intervals $\{[0,\beta[, [\beta,1[\}$, where $0<\beta<1$. Let $n^{(1)}$ denote the connectedness index of u. The frequencies of factors of given length $n \geq n^{(1)}$ of u take at most 5 values. Furthermore, the set of factors of u is stable by mirror image, i.e., if the word $a_1\cdots a_n$ is a factor of the sequence u, then $a_n\cdots a_1$ is also a factor and furthermore, both words have the same frequency.*

Proof. It remains to prove the part of this theorem concerning the stability by mirror image. Assume we are given a fixed positive integer n. Let s_n be the reflection of the circle defined by $s_n\colon x \to \{\beta-(n-1)\alpha-x\}$. We have $s_n(R^{-k}(I_j)) = R^{(-n+1+k)}(I_j)$, for $j=0,1$, following the previous notation. The image of $I(w_1,\ldots,w_n)$ by s_n is $I(w_n,\ldots,w_1)$; they thus have the same length, which gives the result.

REMARK. A study of the topology of the graph of words for a binary coding of an irrational rotation of complexity satisfying ultimately $p(n+1)-p(n)=2$ can be found in [24] or in [46].

8. THE $3d$ DISTANCE THEOREM

Following the idea of the above proof, let us give a combinatorial proof of the $3d$ *distance theorem*.

THE $3d$ DISTANCE THEOREM. *Assume we are given $0<\alpha<1$ irrational, $\gamma_1,\ldots,\gamma_d$ real numbers and $n_1,\ldots,n_d$ positive integers. The points $\{n\alpha+\gamma_i\}$, for $0\leq n<n_i$ and $1\leq i\leq d$, partition the unit circle into at most $n_1+\cdots+n_d$ intervals, having at most $3d$ different lengths.*

Proof. Let us consider a coding of the rotation by angle α under the left-closed and right-open partition of the unit circle bounded by all the points of the form $\{n\alpha + \gamma_i\}$, for $0 \leq n < n_i$ and $1 \leq i \leq d$; let $\beta_0, \ldots, \beta_{p-1}$ denote these consecutive points. The letter associated with the interval $I_k = [\beta_k, \beta_{k+1}[$ has a unique right extension, except when I_k contains points of the form $\{\beta_i - \alpha\}$. Suppose there are $q \geq 2$ points of this form; the associated letter has $q+1$ right extensions. Since there are at most d points of this type, we obtain $p(2) - p(1) \leq d$. We deduce from Theorem 6 that there are at most $3d$ different frequencies for the letters of the coding, i.e., there are at most $3d$ different lengths for the intervals I_k.

REMARK. The start and finish intervals as introduced by Liang in his proof in [37] correspond exactly to the beginning of the branches in the graph of words. Indeed, Liang shows that any interval is associated either with a start point $\{\gamma_i\}$ (i.e., with one extension of a factor having more than one right extension) or with a finish point $\{(n_i - 1)\alpha + \gamma_i\}$ (i.e., with a factor having more than one left extension). Counting the finish and start points defined in [37] (there are $3d$ such points) is equivalent to counting the number of branches in the graph of words.

As in the remark of the previous section, we can consider a coding of the rotation by irrational angle $1-\alpha$ under the partition $\{[\gamma_1, \gamma_2[, \ldots, [\gamma_d, \gamma_1[\}$. For such a coding, the $3d$ distance theorem can be rephrased as follows.

THEOREM 20. *The frequencies of the factors of given length $n \geq n^{(1)}$ of a coding of a rotation by irrational angle under a partition in d intervals take at most $3d$ values, where $n^{(1)}$ denotes the connectedness index.*

ACKNOWLEDGEMENTS. We would like to thank Arnoux, Berstel, Daudé, Dumont, Liardet and Shallit for their bibliographic help. Furthermore, we are greatly indebted to Allouche for his many useful comments and to Ferenczi and Jenkinson who read carefully a previous version of this paper.

REFERENCES

[1] ALESSANDRI, P. Codages de rotations et basses complexités. Université Aix-Marseille II. Thèse, 1996.

[2] ALLOUCHE, J.-P. Sur la complexité des suites infinies. *Bull. Belg. Math. Soc. Simon Stevin 1* (1994), 133–143.