# 5. The recurrence function

is the Sturmian sequence obtained as the coding of the orbit of $-\rho/\alpha$ under the rotation by angle $1/\alpha$, with respect to the partition

$$\{]0, 1-1/\alpha], ]1-1/\alpha, 1]\} .$$

Indeed, if $n = \lfloor \alpha m + \rho \rfloor$, then $\lceil 1/\alpha(n+1) - \rho/\alpha \rceil = m+1 = 1 + \lceil n/\alpha - \rho/\alpha \rceil$, and if $\lfloor \alpha m + \rho \rfloor < n < \lfloor \alpha(m+1) + \rho \rfloor$, then $\lceil 1/\alpha(n+1) - \rho/\alpha \rceil = \lceil n/\alpha - \rho/\alpha \rceil$.

## 5. THE RECURRENCE FUNCTION

Let us deduce now from the three distance and three gap theorems a simple proof of the following result originally due to Morse and Hedlund concerning the recurrence function of a Sturmian sequence (see [40]).

Recall that a sequence $u$ is called *minimal* or *uniformly recurrent* if every factor of $u$ appears infinitely often and with bounded gaps or, equivalently, if for any integer $n$, there exists an integer $m$ such that every factor of $u$ of length $m$ contains every factor of $u$ of length $n$. Note that it is equivalent (see [30]) to the *minimality* of the dynamical system $(\overline{\mathcal{O}(u)}, T)$, i.e., the orbit of every element of $\overline{\mathcal{O}(u)}$ is dense, or equivalently every sequence in the orbit closure of $u$ has the same set of factors as $u$.

The recurrence function $\varphi$ of a minimal sequence $u$ is defined by

$$\varphi(n) = \min\{m \in \mathbf{N} \text{ such that } \forall B \in L_m,\ \forall A \in L_n,\ A \text{ is a factor of } B\} ,$$

where $L_n$ denotes the set of factors of $u$ of length $n$, i.e., $\varphi(n)$ is the size of the smallest window that contains all factors of length $n$ whatever its position in the sequence.

THEOREM 11. *Let $u$ be a Sturmian sequence with angle $\alpha$. Let $(q_k)_{k\in\mathbf{N}}$ denote the sequence of denominators of the convergents of the continued fraction expansion of $\alpha$. The recurrence function $\varphi$ of this sequence satisfies for any non zero integer $n$ :*

$$\varphi(n) = n - 1 + q_k + q_{k-1}, \quad \textit{where} \quad q_{k-1} \le n < q_k .$$

*Proof of Theorem 11.* Let $u \in \{0,1\}^{\mathbf{N}}$ be a Sturmian sequence. There exist a real number $x$ and an irrational number $\alpha$ in $]0,1[$ such that $u_n = 0 \Leftrightarrow \{x + n\alpha\} \in I_0$, with $I_0 = [0,\alpha[$ or $I_0 = ]0,\alpha]$ (see Section 2.1). Let $I_1 = [\alpha, 1[$ (respectively, $]\alpha, 1]$) if $I_0 = [0,\alpha[$ (respectively, $I_0 = ]0,\alpha]$). Let us denote by $R$ the rotation of the circle by angle $\alpha$. Assume we are given a positive integer $n$. We have seen in Section 2.1 that the word $w_1 w_2 \cdots w_n$ defined on $\{0,1\}$ appears in $u$ if and only if

$$I(w_1, \ldots, w_n) = \bigcap_{j=0}^{n-1} R^{-j}(I_{w_{j+1}}) \neq \varnothing .$$

We deduce from this that every Sturmian sequence of angle $\alpha$ has the same factors as $u$ and thus belongs to the orbit closure of $u$. Conversely, each sequence of the orbit closure of $u$ is a Sturmian sequence of angle $\alpha$. Hence the closed orbit of any Sturmian sequence is equal to the set of all Sturmian sequences of the same angle. This implies the minimality of any Sturmian sequence and that Sturmian sequences of the same angle have the same recurrence function; hence we can suppose here that $x = 0$.

Theorem 11 can easily be deduced from the following two lemmata. We omit the proof of Lemma 5, which is straightforward.

LEMMA 4. *Let $\delta_n$ be the smallest length of the nonempty intervals $I(w_1, \ldots, w_n)$, where $w_1, \ldots, w_n$ belong to $\{0,1\}$. Let $l_n$ be the greatest gap between the successive integers $k$ such that $\{k\alpha\} \in [0, \delta_n[$. We have*

$$\varphi(n) = n - 1 + l_n .$$

LEMMA 5. *Let $(q_k)_{k \in \mathbf{N}}$ denote the sequence of denominators of the convergents of the continued fraction expansion of $\alpha$. Let $k$ be an integer such that $q_{k-1} \leq n < q_k$. Then we have*

$$\delta_n = \eta_{k-1} \quad \text{and} \quad l_n = q_k + q_{k-1} .$$

*Proof of Lemma 4.* A set of points is said to visit an interval if one of these points belongs to this interval. By definition of $l_n$, every set of $l_n$ consecutive points of the sequence $(\{k\alpha\})_{k \in \mathbf{N}}$ visits every interval of length $\delta_n$ (see above Remarks). Therefore they visit every nonempty interval of the form $I(w_1, \ldots, w_n)$, by definition of $\delta_n$. Let $B$ be a factor of $u$ of length $n-1+l_n$; there exists an integer $K$ such that $B$ corresponds to the $n-1+l_n$ consecutive points $\{K\alpha\}, \ldots, \{(K+n-1+l_n-1)\alpha\}$. The set of the $l_n$ first points of this sequence of points visits every interval of the form $I(w_1, \ldots, w_n)$, thus $B$ contains every factor of $u$ of length $n$. This implies that $\varphi(n) \leq n-1+l_n$.

By definition of $l_n$ and by density of $(\{k\alpha\})_{k \in \mathbf{N}}$, there exists a sequence of $l_n - 1$ points of the sequence $(\{k\alpha\})_{k \in \mathbf{N}}$ which do not visit an interval of the form $I(w_1, \dots, w_n)$ of length $\delta_n$; therefore, there exists a factor of $u$ of length $l_n - 2 + n$ which does not contain the factor $w_1 \cdots w_n$. This shows that $\varphi(n) \geq n - 1 + l_n$. The lemma is thus proved.

REMARK. Note that in the case of the Fibonacci sequence ($\alpha = \frac{\sqrt{5}-1}{2}$), the recurrence function satisfies, for $F_{k-1} < n \leq F_k$,

$$\varphi(n) = n - 1 + F_{k+1},$$

where $(F_n)_{n \in \mathbf{N}}$ denotes the Fibonacci sequence $F_{n+1} = F_n + F_{n-1}$, with $F_0 = 1$ and $F_1 = 2$.

This result is extended in [13] to the fixed point of the substitution $\sigma$ introduced by Rauzy which generalizes the Fibonacci substitution and is defined by $\sigma(0) = 01$, $\sigma(1) = 02$, $\sigma(2) = 0$.

THEOREM 12. *Let $T_n$ denote the so-called Tribonacci sequence defined as follows: $T_{k+3} = T_{k+2} + T_{k+1} + T_k$, with $T_0 = 0$, $T_1 = 0$, $T_1 = 1$. The recurrence function $\varphi$ of the fixed point beginning with $0$ of the Rauzy substitution satisfies for any positive integer $n$:*

$$\varphi(n) = n - 1 + T_{k+6}, \quad \textit{where} \quad \sum_0^{k+1} T_i < n \leq \sum_0^{k+2} T_i.$$

## 6. HIGHER-DIMENSIONAL GENERALIZATIONS

### 6.1 TWO-DIMENSIONAL GENERALIZATIONS AND BEATTY SEQUENCES

Let us consider now some two-dimensional versions of the three distance and three gap theorems. Such generalizations were introduced by Fraenkel and Holzman in [26] in order to give an upper bound for the number of gaps in the intersection of two Beatty sequences. They first reduce this problem to a two-dimensional version of the three distance theorem, conjectured by Simpson and Holzman and proved by Geelen and Simpson (see [29]). Then they deduce from this theorem a bound for the number of gaps in the intersection of two Beatty sequences, when at least one of the moduli is rational.

Let us first give the two-dimensional version of the three gap theorem introduced by Fraenkel and Holzman. We will use the same notation as in [26]: for any pair of real numbers $(x, y)$, $\{(x, y)\}$ means the equivalence class of $(x, y)$ mod $\mathbf{Z}^2$, i.e., $\{(x, y)\}$ belongs to the torus $\mathbf{T}^2$.