4.3 Beatty sequences

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Let a be the limit of this sequence, $n^{(2)}$ the smallest index such that $k_n = a$, and let

$$b = \sum_{i=0}^{n^{(2)}-1} (k_i - a) .$$

Let $n^{(1)}$ denote the connectedness index of u.

If $n \geq \max(n^{(1)}, n^{(2)})$, then the complexity of the sequence u satisfies

$$p(n) = an + b .$$

REMARKS.

• Note that if $1, \alpha, \beta_1, \ldots, \beta_p$ are rationally independent, then $n^{(2)} = 0$, $b = 0$ and $a = p$.

• Theorem 10 answers the question of the existence of sequences of ultimately affine complexity (for more details, the reader is referred to [1], see also the result of Cassaigne in [11]).

4.3 BEATTY SEQUENCES

The connections between the three gap theorem and the Beatty sequences have been investigated by Fraenkel and Holzman in [26]. Let us recall that a Beatty sequence is a sequence $u(\alpha, \rho) = (u_n)_{n \in \mathbf{N}}$ of the form $u_n = \lfloor \alpha n + \rho \rfloor$, where α and ρ are real numbers such that $\alpha \geq 1$. The number α is called the *modulus* and ρ is called the *residue* or *intercept*. For an impressive bibliography on the subject, we refer the reader to [27] and [54]. Fraenkel and Holzman have noticed in [26] that the three gap theorem answers the question of the gaps in the intersection of a Beatty sequence and an arithmetical sequence $(an + c)_{n \in \mathbf{N}}$, for a a positive integer and c an integer. This result has been obtained independently by Wolff and Pitman in [58]. By intersection of the two Beatty sequences $s = (s_n)_{n \in \mathbf{N}}$ and $t = (t_n)_{n \in \mathbf{N}}$, we mean the strictly increasing sequence u defined as:

$$\{u_n, \ n \in \mathbf{N}\} = \{u, \ \exists k, l \in \mathbf{N} \text{ such that } u = s_k = t_l\} .$$

Hence a gap in the intersection denotes the difference between two distinct elements of the intersection.

Note that Beatty sequences and Sturmian sequences are related: let u be a Beatty sequence of modulus α and residue ρ; the characteristic sequence $(v_n)_{n \in \mathbf{N}}$ of u defined as

$$v_n = 1 \text{ if and only if there exists } m \text{ such that } n = \lfloor \alpha m + \rho \rfloor$$

is the Sturmian sequence obtained as the coding of the orbit of $-\rho/\alpha$ under the rotation by angle $1/\alpha$, with respect to the partition

$$\{]0, 1 - 1/\alpha],]1 - 1/\alpha, 1]\} .$$

Indeed, if $n = \lfloor \alpha m+\rho \rfloor$, then $\lceil 1/\alpha(n+1)-\rho/\alpha \rceil = m+1 = 1+\lceil n/\alpha-\rho/\alpha \rceil$, and if $\lfloor \alpha m+\rho \rfloor < n < \lfloor \alpha(m+1)+\rho \rfloor$, then $\lceil 1/\alpha(n+1)-\rho/\alpha \rceil = \lceil n/\alpha-\rho/\alpha \rceil$.

5. THE RECURRENCE FUNCTION

Let us deduce now from the three distance and three gap theorems a simple proof of the following result originally due to Morse and Hedlund concerning the recurrence function of a Sturmian sequence (see [40]).

Recall that a sequence u is called *minimal* or *uniformly recurrent* if every factor of u appears infinitely often and with bounded gaps or, equivalently, if for any integer n, there exists an integer m such that every factor of u of length m contains every factor of u of length n. Note that it is equivalent (see [30]) to the *minimality* of the dynamical system $(\overline{\mathcal{O}(u)}, T)$, i.e., the orbit of every element of $\overline{\mathcal{O}(u)}$ is dense, or equivalently every sequence in the orbit closure of u has the same set of factors as u.

The recurrence function φ of a minimal sequence u is defined by

$$\varphi(n) = \min \{m \in \mathbf{N} \text{ such that } \forall B \in L_m, \forall A \in L_n, \ A \text{ is a factor of } B\} ,$$

where L_n denotes the set of factors of u of length n, i.e., $\varphi(n)$ is the size of the smallest window that contains all factors of length n whatever its position in the sequence.

THEOREM 11. *Let u be a Sturmian sequence with angle α. Let $(q_k)_{k\in\mathbf{N}}$ denote the sequence of denominators of the convergents of the continued fraction expansion of α. The recurrence function φ of this sequence satisfies for any non zero integer n:*

$$\varphi(n) = n - 1 + q_k + q_{k-1}, \ \textit{where} \ \ q_{k-1} \leq n < q_k .$$

Proof of Theorem 11. Let $u \in \{0,1\}^{\mathbf{N}}$ be a Sturmian sequence. There exist a real number x and an irrational number α in $]0,1[$ such that $u_n = 0 \Leftrightarrow \{x+n\alpha\} \in I_0$, with $I_0 = [0,\alpha[$ or $I_0 =]0,\alpha]$ (see Section 2.1). Let $I_1 = [\alpha, 1[$ (respectively, $]\alpha, 1]$) if $I_0 = [0,\alpha[$ (respectively, $I_0 =]0,\alpha]$). Let us denote by R the rotation of the circle by angle α. Assume we are given