

## 4.3 Beatty sequences

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Let  $a$  be the limit of this sequence,  $n^{(2)}$  the smallest index such that  $k_n = a$ , and let

$$b = \sum_{i=0}^{n^{(2)}-1} (k_i - a).$$

Let  $n^{(1)}$  denote the connectedness index of  $u$ .

If  $n \geq \max(n^{(1)}, n^{(2)})$ , then the complexity of the sequence  $u$  satisfies

$$p(n) = an + b.$$

#### REMARKS.

- Note that if  $1, \alpha, \beta_1, \dots, \beta_p$  are rationally independent, then  $n^{(2)} = 0$ ,  $b = 0$  and  $a = p$ .

- Theorem 10 answers the question of the existence of sequences of ultimately affine complexity (for more details, the reader is referred to [1], see also the result of Cassaigne in [11]).

### 4.3 BEATTY SEQUENCES

The connections between the three gap theorem and the Beatty sequences have been investigated by Fraenkel and Holzman in [26]. Let us recall that a Beatty sequence is a sequence  $u(\alpha, \rho) = (u_n)_{n \in \mathbf{N}}$  of the form  $u_n = \lfloor \alpha n + \rho \rfloor$ , where  $\alpha$  and  $\rho$  are real numbers such that  $\alpha \geq 1$ . The number  $\alpha$  is called the *modulus* and  $\rho$  is called the *residue* or *intercept*. For an impressive bibliography on the subject, we refer the reader to [27] and [54]. Fraenkel and Holzman have noticed in [26] that the three gap theorem answers the question of the gaps in the intersection of a Beatty sequence and an arithmetical sequence  $(an + c)_{n \in \mathbf{N}}$ , for  $a$  a positive integer and  $c$  an integer. This result has been obtained independently by Wolff and Pitman in [58]. By intersection of the two Beatty sequences  $s = (s_n)_{n \in \mathbf{N}}$  and  $t = (t_n)_{n \in \mathbf{N}}$ , we mean the strictly increasing sequence  $u$  defined as:

$$\{u_n, n \in \mathbf{N}\} = \{u, \exists k, l \in \mathbf{N} \text{ such that } u = s_k = t_l\}.$$

Hence a gap in the intersection denotes the difference between two distinct elements of the intersection.

Note that Beatty sequences and Sturmian sequences are related: let  $u$  be a Beatty sequence of modulus  $\alpha$  and residue  $\rho$ ; the characteristic sequence  $(v_n)_{n \in \mathbf{N}}$  of  $u$  defined as

$$v_n = 1 \text{ if and only if there exists } m \text{ such that } n = \lfloor \alpha m + \rho \rfloor$$

is the Sturmian sequence obtained as the coding of the orbit of  $-\rho/\alpha$  under the rotation by angle  $1/\alpha$ , with respect to the partition

$$\{]0, 1 - 1/\alpha], ]1 - 1/\alpha, 1]\} .$$

Indeed, if  $n = \lfloor \alpha m + \rho \rfloor$ , then  $\lceil 1/\alpha(n+1) - \rho/\alpha \rceil = m+1 = 1 + \lceil n/\alpha - \rho/\alpha \rceil$ , and if  $\lfloor \alpha m + \rho \rfloor < n < \lfloor \alpha(m+1) + \rho \rfloor$ , then  $\lceil 1/\alpha(n+1) - \rho/\alpha \rceil = \lceil n/\alpha - \rho/\alpha \rceil$ .

## 5. THE RECURRENCE FUNCTION

Let us deduce now from the three distance and three gap theorems a simple proof of the following result originally due to Morse and Hedlund concerning the recurrence function of a Sturmian sequence (see [40]).

Recall that a sequence  $u$  is called *minimal* or *uniformly recurrent* if every factor of  $u$  appears infinitely often and with bounded gaps or, equivalently, if for any integer  $n$ , there exists an integer  $m$  such that every factor of  $u$  of length  $m$  contains every factor of  $u$  of length  $n$ . Note that it is equivalent (see [30]) to the *minimality* of the dynamical system  $(\overline{\mathcal{O}(u)}, T)$ , i.e., the orbit of every element of  $\overline{\mathcal{O}(u)}$  is dense, or equivalently every sequence in the orbit closure of  $u$  has the same set of factors as  $u$ .

The recurrence function  $\varphi$  of a minimal sequence  $u$  is defined by

$$\varphi(n) = \min \{m \in \mathbf{N} \text{ such that } \forall B \in L_m, \forall A \in L_n, A \text{ is a factor of } B\} ,$$

where  $L_n$  denotes the set of factors of  $u$  of length  $n$ , i.e.,  $\varphi(n)$  is the size of the smallest window that contains all factors of length  $n$  whatever its position in the sequence.

**THEOREM 11.** *Let  $u$  be a Sturmian sequence with angle  $\alpha$ . Let  $(q_k)_{k \in \mathbf{N}}$  denote the sequence of denominators of the convergents of the continued fraction expansion of  $\alpha$ . The recurrence function  $\varphi$  of this sequence satisfies for any non zero integer  $n$ :*

$$\varphi(n) = n - 1 + q_k + q_{k-1}, \text{ where } q_{k-1} \leq n < q_k .$$

*Proof of Theorem 11.* Let  $u \in \{0, 1\}^{\mathbf{N}}$  be a Sturmian sequence. There exist a real number  $x$  and an irrational number  $\alpha$  in  $]0, 1[$  such that  $u_n = 0 \Leftrightarrow \{x + n\alpha\} \in I_0$ , with  $I_0 = [0, \alpha[$  or  $I_0 = ]0, \alpha]$  (see Section 2.1). Let  $I_1 = [\alpha, 1[$  (respectively,  $] \alpha, 1]$ ) if  $I_0 = [0, \alpha[$  (respectively,  $I_0 = ]0, \alpha]$ ). Let us denote by  $R$  the rotation of the circle by angle  $\alpha$ . Assume we are given