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Objektyp: **Article**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **44 (1998)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **20.09.2024**

Persistenter Link: <https://doi.org/10.5169/seals-63892>

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A PARTICULAR CASE OF
DIRICHLET'S THEOREM ON ARITHMETIC PROGRESSIONS

by Nairi SEDRAKIAN and John STEINIG

Dirichlet's theorem on primes in an arithmetic progression states that if a and m are relatively prime integers, there exist infinitely many primes p such that $p \equiv a \pmod{m}$. We give here an elementary proof of the case in which $a = 1$.

We use the following notation. If x_1, \dots, x_r are positive integers, with $r \geq 2$, (x_1, \dots, x_r) denotes their greatest common divisor and $[x_1, \dots, x_r]$ their least common multiple. For $r = 1$, we set $(x_1) := x_1$ and $[x_1] := x_1$.

The proof rests on three lemmas.

LEMMA 1. *If m and a_1, \dots, a_r are integers, with $m > 1$ and $a_i \geq 1$ for $1 \leq i \leq r$, then*

$$(1) \quad (m^{a_1} - 1, \dots, m^{a_r} - 1) = m^{(a_1, \dots, a_r)} - 1.$$

Proof. The case $r = 1$ is trivial. The case $r = 2$ can be established by computing $(m^{a_1} - 1, m^{a_2} - 1)$ with the euclidean algorithm; the computation runs parallel to that of (a_1, a_2) . One can then continue by induction, using the associative property

$$(x_1, \dots, x_r) = ((x_1, \dots, x_{r-1}), x_r).$$

(For a different proof of the case $r = 2$, see [8], p. 26.)

LEMMA 2. If x_1, \dots, x_r are positive integers, then

$$(2) \quad [x_1, \dots, x_r] = \frac{\prod_i (x_i) \prod_{i < j < k} (x_i, x_j, x_k) \cdots}{\prod_{i < j} (x_i, x_j) \prod_{i < j < k < \ell} (x_i, x_j, x_k, x_\ell) \cdots},$$

where the numerator on the right hand side is the product of the gcd's of x_1, \dots, x_r , taken n at a time for odd $n = 1, 3, \dots$; the denominator is the product of the gcd's of x_1, \dots, x_r , taken n at a time for even $n = 2, 4, \dots$. There are 2^{r-1} factors in the numerator and $2^{r-1} - 1$ in the denominator.

Proof. The case $r = 1$ is trivial. For $r = 2$, identity (2) is the familiar

$$(3) \quad [x_1, x_2] = \frac{x_1 x_2}{(x_1, x_2)}.$$

One can continue by induction, using (3) and the associative and distributive properties

$$[x_1, \dots, x_r] = [[x_1, \dots, x_{r-1}], x_r],$$

respectively

$$([x_1, \dots, x_{r-1}], x_r) = [(x_1, x_r), \dots, (x_{r-1}, x_r)].$$

(Identity (2) is due to V.-A. Le Besgue ([3], pp.51–53), whose proof consists in showing that any prime divides both sides of (2) to the same power.)

LEMMA 3. Let m be an integer, $m > 1$; let p_1, \dots, p_r be distinct primes which divide m . Then

$$(4) \quad [m^{m/p_1} - 1, \dots, m^{m/p_r} - 1] < m^m - 1.$$

Proof. Since $m^m - 1$ is divisible by each integer $m^{m/p_i} - 1$ ($i = 1, \dots, r$), it is divisible by their least common multiple. Hence (4) will be proved if we can show that

$$(5) \quad [m^{m/p_1} - 1, \dots, m^{m/p_r} - 1] = m^m - 1$$

is impossible. To this end, we rewrite the left hand side of (5) by setting $x_i = m^{m/p_i} - 1$ in Lemma 2, and then apply Lemma 1 to the gcd's which occur. Since p_1, \dots, p_r are distinct primes, we have

$$(x_{i_1}, \dots, x_{i_t}) = m^{m/p_{i_1} \cdots p_{i_t}} - 1 \quad \text{if } 1 \leq i_1 < \cdots < i_t \leq r.$$

This will bring (5) to the form

$$(6) \quad \prod_{j=1}^k (m^{n_j} - 1) = \prod_{j=k+1}^{2k} (m^{n_j} - 1),$$

with $k = 2^{r-1}$ and $n_1 = \frac{m}{p_1 \cdots p_r} < n_j \quad (j \geq 2)$.

But (6) would imply that

$$(-1)^{k-1} (m^{n_1} - 1) \equiv (-1)^k \pmod{m^{n_1+1}},$$

that is, $m^{n_1+1} \mid m^{n_1}$; this is impossible, since $m > 1$. This concludes the proof.

We can now prove the

THEOREM. *Let m be an integer, $m > 1$. There exist infinitely many primes p such that $p \equiv 1 \pmod{m}$.*

Proof. By a familiar argument [10], it suffices to prove the existence, for each $m > 1$, of at least one prime $p \equiv 1 \pmod{m}$. (If $p_1 \equiv 1 \pmod{m}$ and $p_2 \equiv 1 \pmod{p_1 m}$, then $p_2 \equiv 1 \pmod{m}$ and $p_2 \geq p_1 m + 1 > p_1$.)

Now let m be an integer, $m > 1$, and let p_1, \dots, p_s be its distinct prime divisors. Define the integer N by

$$(7) \quad N := \frac{m^m - 1}{[m^{m/p_1} - 1, \dots, m^{m/p_s} - 1]}.$$

Then $N > 1$ by Lemma 3. Let q be any prime divisor of N ; we shall show that

$$(8) \quad q \equiv 1 \pmod{m}.$$

Since $q \mid N$, we have

$$(9) \quad q \mid \frac{m^m - 1}{m^{m/p_i} - 1} \quad \text{for } i = 1, \dots, s$$

and

$$(10) \quad q \mid m^m - 1.$$

It follows from (10) that q does not divide m , whence

$$(11) \quad q \mid m^{q-1} - 1.$$

By (10), (11) and Lemma 1,

$$(12) \quad q \mid m^{(m, q-1)} - 1.$$

Suppose now that (8) does not hold. Then $(m, q-1) \mid \frac{m}{p_i}$ for some i , $1 \leq i \leq s$, whence by (12),

$$(13) \quad q \mid m^{m/p_i} - 1$$

and therefore

$$(14) \quad \frac{m^m - 1}{m^{m/p_i} - 1} = \sum_{\nu=0}^{p_i-1} (m^{m/p_i})^\nu \equiv p_i \pmod{q}.$$

But (14) is impossible, for with (9) it implies that $p_i = q$, contradicting the fact that q does not divide m . This concludes the proof of the theorem.

REMARK. Several elementary proofs of this special case of Dirichlet's theorem are known; see [1], [2, §11.3], [4, §48], [5], [6], [7, §6.1A], [8, Ch. 6,5], [9], [10] and the references in [7, pp.241–245]. They involve, more or less explicitly, the cyclotomic polynomials, say $\Phi_n(x)$. Although the proof we have given here does not require any knowledge of these polynomials, the integer N defined in (7) is in fact equal to $\Phi_m(m)$, as can be seen with Lemmas 1 and 2 and the identity [2, p.181]

$$\Phi_n(x) = \prod_{d|n} (x^{n/d} - 1)^{\mu(d)},$$

where μ is the Möbius function (see also [4], §46).

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(Reçu le 30 juin 1997; version révisée reçue le 30 avril 1998)

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