4. The work of Frobenius

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Let $X_i = X_{g_i}$, $1 \le i \le 8$. Dedekind computed

$$\Theta(Q_8) = \Phi_1\Phi_2\Phi_3\Phi_4\Phi_5^2,$$

where

$$\begin{aligned}
\Phi_1 &= X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7 + X_8, \\
\Phi_2 &= X_1 + X_2 + X_3 + X_4 - X_5 - X_6 - X_7 - X_8, \\
\Phi_3 &= X_1 + X_2 - X_3 - X_4 + X_5 + X_6 - X_7 - X_8, \\
\Phi_4 &= X_1 + X_2 - X_3 - X_4 - X_5 - X_6 + X_7 + X_8, \\
\Phi_5 &= \sum X_i^2 - 2X_1X_2 - 2X_3X_4 - 2X_5X_6 - 2X_7X_8 \\
&= (X_1 - X_2)^2 + (X_3 - X_4)^2 + (X_5 - X_6)^2 + (X_7 - X_8)^2.
\end{aligned}$$

Only Φ_5 is not linear, and it is irreducible over $\mathbf{C}$. There is an obvious "hypercomplex" number system over which Φ_5 becomes a product of linear factors, namely the quaternions $\mathbf{H}$ (although $\mathbf{C}$ is not in its center).

In general, Dedekind wanted to find a hypercomplex number system over which $\Theta(G)$ factors linearly and understand how the structure of G is reflected in such a hypercomplex system. Ten years later, in 1896, Dedekind classified the finite groups all of whose subgroups are normal (Hamiltonian groups), and in a letter to Frobenius where he wrote about this result [10, pp. 420–421], Dedekind mentioned the group determinant, explained how it factors in the abelian case, and suggested Frobenius think about the nonabelian case. It is the question of factoring the group determinant of an arbitrary finite group that gave rise to representation theory by Frobenius, though other algebraic developments in the late 19th century were also heading in this direction [25].

4. The work of Frobenius

Frobenius felt the interesting problem was not finding a hypercomplex number system where $\Theta(G)$ becomes a product of linear factors, but finding the irreducible factors of $\Theta(G)$ over the complex numbers, whether or not they are linear. His solution to this problem appeared in [22], and depended on the papers [20] and [21], where he established the needed facts about commuting matrices and characters of finite groups.

Frobenius begins [21] by recalling previous uses of characters in number theory. Here is how the paper starts:

> «When he proved that every linear function of one variable represents infinitely many prime numbers if its coefficients are coprime integers, Dirichlet used for the first time certain systems of roots of unity, which also appear when one treats the closely related problem of the number of ideal classes in cyclotomic fields [...]»

Establishing Dirichlet's theorem on primes in the arithmetic progression $m + nj$ ($j \in \mathbf{N}$) and the class number formula for the cyclotomic field $\mathbf{Q}(\zeta_n)$ involve not only the characters of $(\mathbf{Z}/n\mathbf{Z})^\times$, but also something Frobenius did not explicitly refer to: L-functions of these characters. This was about thirty years before Artin [1] introduced L-functions of the characters Frobenius introduced in [21].

Towards the end of the introduction to [21] is a prescient comment:

> «In April of this year, Dedekind gave me an exercise ... [whose] solution, which I hope to be able to present soon, led me to a generalization of the notion of a character to arbitrary finite groups. I want to develop this notion here since I believe that by its introduction, group theory should undergo a major advancement and enrichment.»

We begin our analysis of the factorization of $\Theta(G)$ by writing down one factor that is always present and indicating how to normalize the factorization. Since each row of the matrix $(X_{gh^{-1}})$ contains the sequence $\{X_g\}$ in some order, adding all the columns to a fixed column shows $\Theta(G)$ is divisible by $\sum_{g \in G} X_g$. This observation, for cyclic G, was made by Catalan when he first introduced circulants.

Since $\Theta(G)$ is homogeneous of degree $n = \#G$, its irreducible factors are also homogeneous. If we set the variables X_g for $g \neq e$ equal to 0, then $\Theta(G)$ becomes the polynomial X_e^n. Therefore we fix a definite factorization of $\Theta(G)$ into irreducibles by requiring the irreducible factors to be monic in X_e. It will turn out that this is Frobenius' factorization of the group determinant of G.

What are irreducible factors of $\Theta(G)$ besides $\sum X_g$? For each character $\chi\colon G \to \mathbf{C}^\times$ we have a factor $\sum_g \chi(g) X_g$, proven just as in the proof of Theorem 2. This accounts for $\#G/[G,G]$ factors, which leaves more factors to determine for nonabelian G.

In only a few months Frobenius solved this problem. Letting s denote the number of conjugacy classes of G, Frobenius proved $\Theta(G)$ has s (homogeneous) irreducible factors that are monic in X_e, each one having degree equal to its multiplicity in the factorization of $\Theta(G)$. That is,

$$\Theta(G) = \prod_{i=1}^{s} \Phi_i^{f_i},$$

where Φ_i is homogeneous irreducible and f_i is the degree of Φ_i. He also [22, Sect. 12] proved $f_i \mid n$. (Taking degrees of both sides, we get $n = \sum_i f_i^2$, which should look familiar from representation theory. We'll see later that the f_i's are the degrees of the irreducible complex representations of G.) His study of this problem led him to introduce for the first time the notion of a character of a finite nonabelian group, which he defined as a conjugacy class function related to the number of solutions of the equation $ab = c$ where a, b, c run over elements in three conjugacy classes. For a description of this method, see [8, pp. 218–219] or [32, pp. 367–368]. His original notion of character only referred to irreducible ones. The following year would see Frobenius interpret characters as trace functions [23, p. 954]. The basic properties of irreducible characters, such as the orthogonality relations, were first proved without representations. A treatment in English of the group determinant and characters without representation theory was given by Dickson in his 1902 exposition [11] of Frobenius' work.

Rather than go through all of Frobenius' original proof of the factorization of $\Theta(G)$, which did not use representations, we will invoke representation theory in the next section to explain its decomposition.

However, to give a flavor of how Frobenius analyzed the group determinant, we prove a property of its irreducible factors by his techniques (Theorem 3 below). Recall $n = \#G$. We will abbreviate $\Theta(G)$ as Θ.

LEMMA 1. *The adjoint of the group matrix* $(X_{gh^{-1}})$ *has* (g, h) *entry* $(1/n)\,\partial\Theta/\partial X_{hg^{-1}}$.

Proof. Let D be the determinant of a matrix $(a_{g,h})$ doubly indexed by G and having independent entries, so D is a polynomial in $\mathbf{Z}[a_{g,h}]$. The adjoint of the matrix $(a_{g,h})$ is $(\partial D/\partial a_{h,g})$.

Let $\varphi\colon \mathbf{Z}[a_{g,h}] \to \mathbf{Z}[X_r]$ be the ring homomorphism where $\varphi(a_{g,h}) = X_{gh^{-1}}$. So $\varphi(D) = \Theta$, the group determinant. We want to show

$$\varphi\left(\frac{\partial D}{\partial a_{h,g}}\right) = \frac{1}{n}\frac{\partial\Theta}{\partial X_{hg^{-1}}}.$$

By the chain rule, or checking on monomials, for all f in $\mathbf{Z}[a_{g,h}]$ and r in G

$$\frac{\partial\varphi(f)}{\partial X_r} = \sum_{gh^{-1}=r} \varphi\left(\frac{\partial f}{\partial a_{g,h}}\right) = \sum_{k\in G} \varphi\left(\frac{\partial f}{\partial a_{g_0k,h_0k}}\right),$$

where (g_0, h_0) is any pair with $g_0h_0^{-1} = r$.

Let ψ_k be the ring automorphism of $\mathbf{Z}[a_{g,h}]$ where $\psi_k(a_{g,h}) = a_{gk,hk}$. Then $\varphi\psi_k = \varphi$, so

$$\frac{\partial\varphi(f)}{\partial X_r} = \sum_{k\in G} \varphi\psi_k\left(\frac{\partial\psi_k^{-1}f}{\partial a_{g_0,h_0}}\right) = \sum_{k\in G} \varphi\left(\frac{\partial\psi_k^{-1}f}{\partial a_{g_0,h_0}}\right) = \sum_{k\in G} \varphi\left(\frac{\partial\psi_k f}{\partial a_{g_0,h_0}}\right).$$

Now set $f = D$, and note $\psi_k(D) = D$. $\square$

Letting $Y_r = (1/n)\,\partial\Theta/\partial X_{r^{-1}}$, we see the adjoint of $(X_{gh^{-1}})$ has the form $(Y_{gh^{-1}})$.

Polynomials in the X_g's can be viewed as functions on matrices of the form $x = (x_{gh^{-1}})$ or as functions on elements of the group algebra $x = \sum_g x_g g$. For example, viewing the group determinant Θ as such a function, it is multiplicative: $\Theta(xy) = \Theta(x)\,\Theta(y)$. The element xy has g-coordinate $\sum_{ab=g} x_a y_b$.

The next theorem, which appeared in [22, Sect. 1], shows the multiplicative property of Θ passes to its irreducible factors, and in fact characterizes them.

THEOREM 3. *Let Φ be a homogeneous irreducible polynomial in the variables X_g. Then $\Phi(xy) = \Phi(x)\,\Phi(y)$ if and only if Φ is monic in X_e and is a factor of Θ.*

Proof. First we assume Φ is monic in X_e and is a factor of Θ.

Choose indeterminates $\{X_g\}$ and $\{Y_g\}$. Let $Z_g = \sum_{ab=g} X_a Y_b$, so in $\mathbf{C}[X_g, Y_h]$ we have $\Theta(Z) = \Theta(X)\,\Theta(Y)$. Since $\Phi(Z) \mid \Theta(Z)$, $\Phi(Z) = \Lambda(X)\,M(Y)$ for some polynomials Λ in the X's and M in the Y's. Set $Y_e = 1$ and $Y_g = 0$ for $g \neq e$. We get $\Phi(X) = \Lambda(X)M(1,0,0,\dots)$. Similarly, $\Phi(Y) = \Lambda(1,0,0,\dots)M(Y)$. Therefore

$$\Phi(X)\,\Phi(Y) = \Phi(Z)\,\Lambda(1,0,0,\dots)\,M(1,0,0,\dots) = \Phi(XY)\,\Phi(1,0,0,\dots).$$

Since Φ is homogeneous and monic in X_e, $\Phi(1,0,0,\dots) = 1$.

Now assume Φ is multiplicative. Since Φ is homogeneous, we have $\Phi(X_e,0,0,\dots) = cX_e^d$. Letting $Y_e = 1$ and $Y_g = 0$ for $g \neq e$, we have $\Phi(X) = \Phi(X)\Phi(1,0,0,\dots)$, so $\Phi(1,0,0,\dots) = c = 1$, hence Φ is monic in X_e.

By Lemma 1, we can write the adjoint matrix of $(X_{gh^{-1}})$ in the form $(Y_{gh^{-1}})$. For this choice of the Y's, we have in $\mathbf{C}[X_g]$ that $\Phi(X)\,\Phi(Y) = \Phi(\Theta,0,0,\dots) = \Theta^d$, so $\Phi \mid \Theta$. $\square$

The "if" direction did not need Φ to be irreducible. It can also be removed as a hypothesis in the "only if" direction by weakening the conclusion to Φ dividing a power of Θ.

Theorem 3 allowed Frobenius to establish a conjecture of Dedekind [10, p. 422], which said that the linear factors of Θ, monic in X_e, are related to the characters of the abelian group $G/[G,G]$. More precisely, Frobenius showed the linear factors of Θ, monic in X_e, are exactly the polynomials $\sum_g \chi(g)X_g$, where $\chi\colon G \to \mathbf{C}^\times$ is a character, and each such linear factor arises exactly once in the factorization of Θ. (Since we already showed such polynomials are factors, only the "if" direction of Theorem 3 is needed and therefore Lemma 1 is not required for this.) The reader is referred to the paper of Frobenius [22, Sect. 2] or Dickson [11, Sect. 6] for details of this argument.

It is of interest to see what is mentioned about the group determinant in Thomas Muir's *The Theory of Determinants in the Historical Order of Development*, which aimed to describe all developments in the subject up until 1900. In the preface to the final volume, Muir expresses the hope that "little matter of any serious importance has been passed over that was needed for this History." There are many references to the circulant, one to Dedekind's calculation of $\Theta(S_3)$, but there is no mention of any work on the group determinant by Frobenius. However, his List of Writings in the 1907 *Quart. J. Pure Appl. Math.* shows he was aware of such papers.

5. Factoring the group determinant by representation theory

We now use representation theory to completely factor the group determinant. As in the second proof of Theorem 2, let's compute the matrix for left multiplication in $\mathbf{C}[G]$ by an element $\sum a_g g$, with respect to the basis G of $\mathbf{C}[G]$. Since

$$\Big(\sum_g a_g g\Big)h = \sum_g a_{gh^{-1}} g\,,$$

the matrix for left multiplication by $\sum a_g g$ is $(a_{gh^{-1}})$. Hence

$$\det(a_{gh^{-1}}) = \mathrm{N}_{\mathbf{C}[G]/\mathbf{C}}\Big(\sum_g a_g g\Big)\,.$$

Since $\mathbf{C}[G]$ decomposes into a product of matrix algebras, this norm will decompose into a product of determinants. More specifically, let $\{(\rho, V_\rho)\}$ be a full set of mutually nonisomorphic irreducible representations of G (over the complex numbers). Then the map