

§3. Invariants of links

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§3. INVARIANTS OF LINKS

In this section, we give a graphical definition of the HOMFLY polynomial of oriented links. For an oriented link diagram D , we define $\langle D \rangle_n$ by the following.

$$\left\langle \begin{array}{c} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{array} \right\rangle_n = q^{1/2} \left\langle \begin{array}{c} \uparrow \\ \uparrow \end{array} \right\rangle_n - \left\langle \begin{array}{c} \nearrow \quad \nearrow \\ \searrow \quad \searrow \\ \nearrow \quad \nearrow \\ \searrow \quad \searrow \end{array} \right\rangle_n$$

and

$$\left\langle \begin{array}{c} \nearrow \\ \searrow \\ \searrow \\ \nearrow \end{array} \right\rangle_n = q^{-1/2} \left\langle \begin{array}{c} \uparrow \\ \uparrow \end{array} \right\rangle_n - \left\langle \begin{array}{c} \nearrow \quad \nearrow \\ \searrow \quad \searrow \\ \searrow \quad \searrow \\ \nearrow \quad \nearrow \end{array} \right\rangle_n.$$

Then we have

THEOREM 3.1. $\langle D \rangle_n$ is invariant under the Reidemeister moves II and III.

Proof. From Lemma 2.2, we have

$$\begin{aligned} \left\langle \begin{array}{c} \nearrow \\ \searrow \\ \searrow \\ \nearrow \end{array} \right\rangle_n &= \left\langle \begin{array}{c} \uparrow \\ \uparrow \end{array} \right\rangle_n - q^{1/2} \left\langle \begin{array}{c} \nearrow \quad \nearrow \\ \searrow \quad \searrow \\ \nearrow \quad \nearrow \\ \searrow \quad \searrow \end{array} \right\rangle_n \\ &\quad - q^{-1/2} \left\langle \begin{array}{c} \nearrow \quad \nearrow \\ \searrow \quad \searrow \\ \searrow \quad \searrow \\ \nearrow \quad \nearrow \end{array} \right\rangle_n + \left\langle \begin{array}{c} \nearrow \quad \nearrow \\ \searrow \quad \searrow \\ \nearrow \quad \nearrow \\ \searrow \quad \searrow \end{array} \right\rangle_n \\ &= \left\langle \begin{array}{c} \uparrow \\ \uparrow \end{array} \right\rangle_n + (-q^{1/2} - q^{-1/2} + [2]) \left\langle \begin{array}{c} \nearrow \quad \nearrow \\ \searrow \quad \searrow \\ \nearrow \quad \nearrow \\ \searrow \quad \searrow \end{array} \right\rangle_n = \left\langle \begin{array}{c} \uparrow \\ \uparrow \end{array} \right\rangle_n. \end{aligned}$$

We also have

$$\begin{aligned}
 \left\langle \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right\rangle_n &= \left\langle \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right\rangle_n - q^{1/2} \left\langle \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} \right\rangle_n \\
 &\quad - q^{-1/2} \left\langle \begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array} \right\rangle_n + \left\langle \begin{array}{c} \text{Diagram 9} \\ \text{Diagram 10} \end{array} \right\rangle_n \\
 &= ([n] - q^{1/2}[n-1] - q^{-1/2}[n-1] + [n-2]) \left\langle \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right\rangle_n + \left\langle \begin{array}{c} \text{Diagram 11} \\ \text{Diagram 12} \end{array} \right\rangle_n \\
 &= \left\langle \begin{array}{c} \text{Diagram 13} \\ \text{Diagram 14} \end{array} \right\rangle_n
 \end{aligned}$$

from Lemmas 2.1, 2.3 and 2.4 and so $\langle D \rangle_n$ is invariant under the Reidemeister move II. Next we prove the invariance under the Reidemeister move III. Since we have

$$\left\langle \begin{array}{c} \text{Diagram 15} \\ \text{Diagram 16} \end{array} \right\rangle_n = q^{1/2} \left\langle \begin{array}{c} \text{Diagram 17} \\ \text{Diagram 18} \end{array} \right\rangle_n - \left\langle \begin{array}{c} \text{Diagram 19} \\ \text{Diagram 20} \end{array} \right\rangle_n$$

and

$$\left\langle \begin{array}{c} \text{Diagram 21} \\ \text{Diagram 22} \end{array} \right\rangle_n = q^{1/2} \left\langle \begin{array}{c} \text{Diagram 23} \\ \text{Diagram 24} \end{array} \right\rangle_n - \left\langle \begin{array}{c} \text{Diagram 25} \\ \text{Diagram 26} \end{array} \right\rangle_n,$$

The figure consists of two diagrams separated by an equals sign. Each diagram shows a vertical line with a loop. In the left diagram, the top of the loop is labeled '1' and the bottom is labeled '2'. In the right diagram, the top of the loop is labeled '1' and the bottom is labeled '1'. The diagrams are separated by an equals sign.

Figure 1 shows two Feynman diagrams representing the two-point function. The left diagram is multiplied by $-q^{1/2}$. Both diagrams show a central vertex with two external lines (left and right) and two internal lines (top and bottom). The internal lines are further subdivided into a network of vertices and lines, with labels 1 and 2 indicating different types of lines or vertices. The right diagram is added to the left one.

$$= - \left\langle \begin{array}{c} 1 \uparrow \\ \diagup \quad \diagdown \end{array} \right\rangle + \left\langle \begin{array}{c} 1 \uparrow \quad 1 \uparrow \\ \diagdown \quad \diagup \\ 2 \uparrow \\ \diagup \quad \diagdown \\ 1 \uparrow \quad 1 \uparrow \end{array} \right\rangle_n - q^{1/2} \left\langle \begin{array}{c} 1 \uparrow \quad 1 \uparrow \\ \diagdown \quad \diagup \\ 2 \uparrow \\ \diagdown \quad \diagup \\ 1 \uparrow \quad 1 \uparrow \\ \diagdown \quad \diagup \\ 1 \uparrow \quad 1 \uparrow \end{array} \right\rangle_n + \left\langle \begin{array}{c} 1 \uparrow \quad 1 \uparrow \\ \diagdown \quad \diagup \\ 2 \uparrow \quad 1 \uparrow \\ \diagdown \quad \diagup \\ 1 \uparrow \quad 1 \uparrow \\ \diagdown \quad \diagup \\ 1 \uparrow \quad 1 \uparrow \end{array} \right\rangle_n$$

Figure 1 shows two diagrams representing elements of the algebra A_n . The left diagram is a tree with root 3, children 1 and 2, and grandchildren 1, 1, 1, 1, 1, 1. The right diagram is a tree with root 2, children 1 and 1, and grandchildren 1, 1, 1, 1, 1, 1. The diagrams are separated by an equals sign and a minus sign.

from Lemmas 2.2 and 2.5 and

The diagrammatic proof shows the following steps:

- Initial diagram: A link with four strands. The top two strands are labeled 1 and 2, and the bottom two strands are labeled 1 and 2. The strands are connected by crossings.
- First transformation: The diagram is simplified using the relation $q^2 = q + q^{-1}$. The result is a sum of two diagrams, each with a different crossing configuration.
- Second transformation: The diagrams are further simplified using the relation $q^2 = q + q^{-1}$. The result is a sum of two diagrams, each with a different crossing configuration.
- Final result: The diagram is simplified to a sum of two diagrams, each with a different crossing configuration.

$$= \left\langle \begin{array}{c} \text{Diagram 1} \end{array} \right\rangle_n - q^{1/2} \left\langle \begin{array}{c} \text{Diagram 2} \end{array} \right\rangle_n,$$

we have the required formula from Lemma 2.6.

Since the Reidemeister move III with other types of orientations can be obtained from the Reidemeister moves II and III described above (see [12]), the proof is complete. $\square$

If we define $P_n(D) = q^{(n/2)(-w(D))} \langle D \rangle_n$ with $w(D)$ the writhe (the algebraic sum of the crossings) of D , then we have

THEOREM 3.2. $P_n(D)$ is invariant under the Reidemeister moves I, II, and III and satisfies the following skein relation.

$$q^{n/2} P_n(D_+) - q^{-n/2} P_n(D_-) = (q^{1/2} - q^{-1/2}) P_n(D_0),$$

where D_+ , D_- and D_0 are identical link diagrams except near a crossing as described in Figure 3.1.

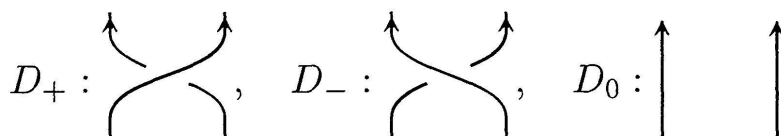


FIGURE 3.1
skein triple

The proof of this theorem follows immediately from the following lemma.

LEMMA 3.3. $\left\langle \begin{array}{c} \text{Diagram 1} \end{array} \right\rangle_n = q^{n/2} \left\langle \begin{array}{c} \text{Diagram 2} \end{array} \right\rangle_n$

and

$$\left\langle \begin{array}{c} \text{Diagram 1} \end{array} \right\rangle_n = q^{-n/2} \left\langle \begin{array}{c} \text{Diagram 2} \end{array} \right\rangle_n.$$

Proof. From Lemmas 2.1 and 2.3, we have

$$(3.1) \quad \left\langle \begin{array}{c} \uparrow \\ \bigcirc \end{array} \right\rangle_n = q^{1/2} \left\langle \begin{array}{c} 1 \uparrow \\ \bigcirc \\ 1 \end{array} \right\rangle_n - \left\langle \begin{array}{c} 1 \uparrow \\ 2 \bigcirc 1 \\ 1 \end{array} \right\rangle_n$$

$$(3.2) \quad = (q^{1/2}[n] - [n-1]) \left\langle \begin{array}{c} \uparrow 1 \\ \end{array} \right\rangle_n.$$

Since

$$q^{1/2}[n] - [n-1] = q^{n/2},$$

the first equality follows. The second equality follows similarly and the proof is complete. $\square$

Therefore P_n defines a link invariant and so we can put $P_n(L) = P_n(D)$ for the link L presented by D . Then P_n is a version of the HOMFLY polynomial [1], [13].

REMARK 3.4. If we define $[D]_n$ to be

$$\left[\begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} \right]_n = Aq^{1/2} \left[\begin{array}{c} 1 \uparrow \\ \end{array} \right]_n - A \left[\begin{array}{c} 1 \nearrow 1 \\ 2 \nwarrow 1 \end{array} \right]_n$$

and

$$\left[\begin{array}{c} \nwarrow \nearrow \\ \nearrow \nwarrow \end{array} \right]_n = A^{-1}q^{-1/2} \left[\begin{array}{c} 1 \uparrow \\ \end{array} \right]_n - A^{-1} \left[\begin{array}{c} 1 \nearrow 1 \\ 2 \nwarrow 1 \end{array} \right]_n$$

with A an indeterminate, we also have a framed link invariant.

When $n = 2$ and $A = q^{-1/4}$, we have a version of the Kauffman bracket naturally defined from representation theory of $U_q(\mathfrak{sl}(2, \mathbb{C}))$ [6, Theorem 4.3].

For $n = 3$ we have G. Kuperberg's recursive formula [8] as follows. Consider an oriented, trivalent, plane graph with flow less than or equal to three. If we reverse the orientations of all the edges with flow two, remove all the edges with flow three, and forget the flow, then we have a trivalent graph without flow such that at every vertex all the edges are in or out. Putting $A = q^{-1/6}$ and replacing q with q^{-1} , we have Kuperberg's formula. (This corresponds to the fact that the two-fold anti-symmetric tensor of the vector representation of $SU(3)$ is isomorphic to its dual.) See also [12].