1. Introduction

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A DYNAMICAL SYSTEMS APPROACH TO BIRKHOFF'S THEOREM

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ABSTRACT. We present a new proof of Birkhoff's classical theorem that an embedded homotopically nontrivial circle, which is invariant under a monotone twist map on $\mathbf{S}^1 \times \mathbf{R}$, must be the graph of a Lipschitz function.

1. INTRODUCTION

Consider the two-dimensional cylinder $\mathbf{S}^1 \times \mathbf{R} \cong \mathbf{R}/\mathbf{Z} \times \mathbf{R}$, respectively its universal cover $\mathbf{R}^2$ with coordinates x, y. A diffeomorphism $\phi \colon \mathbf{S}^1 \times \mathbf{R} \to \mathbf{S}^1 \times \mathbf{R}$ is called a monotone twist mapping if it is area-preserving and satisfies the monotone twist condition $\partial(\pi_x \circ \phi)/\partial y \neq 0$, where π_x denotes the projection onto the first coordinate. This means, in particular, that (pre-)images of verticals under any lift of ϕ are graphs over the x-axis.

The twist condition is not as artificial as it might seem. Monotone twist mappings appear in a variety of situations, often unexpected and only discovered by clever coordinate choices. In the following, we give a few examples. The reader may consult [LCa, MF, Mo1, Mo2] for more detailed information and further references.

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EXAMPLE 1. The simplest examples of monotone twist mappings are integrable ones which, by definition, preserve the y-coordinate. If $\phi\colon (x,y) \mapsto (x+f(x,y),y)$ is area-preserving, i.e. $\phi^*(dx \wedge dy) = dx \wedge dy$, then $f = f(y)$; the monotone twist condition is equivalent to $f'(y) \neq 0$. Hence any integrable monotone twist map is of the form

$$\phi\colon (x,y) \mapsto (x+f(y),y)$$

with some monotone function f. It "twists" the invariant curves $\mathbf{R} \times \{y\}$ in the sense that the angle of rotation on these curves grows or decreases with y at least by some fixed rate δ.

EXAMPLE 2. In some sense the "simplest" non-integrable monotone twist map is the so-called standard map

$$\phi\colon (x,y) \mapsto \left(x+y+\frac{k}{2\pi}\sin 2\pi x,\, y+\frac{k}{2\pi}\sin 2\pi x\right)$$

where $k \geq 0$ is a parameter. This map has been the subject of extensive analytical and numerical studies. Famous pictures illustrate the transition from integrability ($k=0$) to "chaos" ($k \approx 10$).

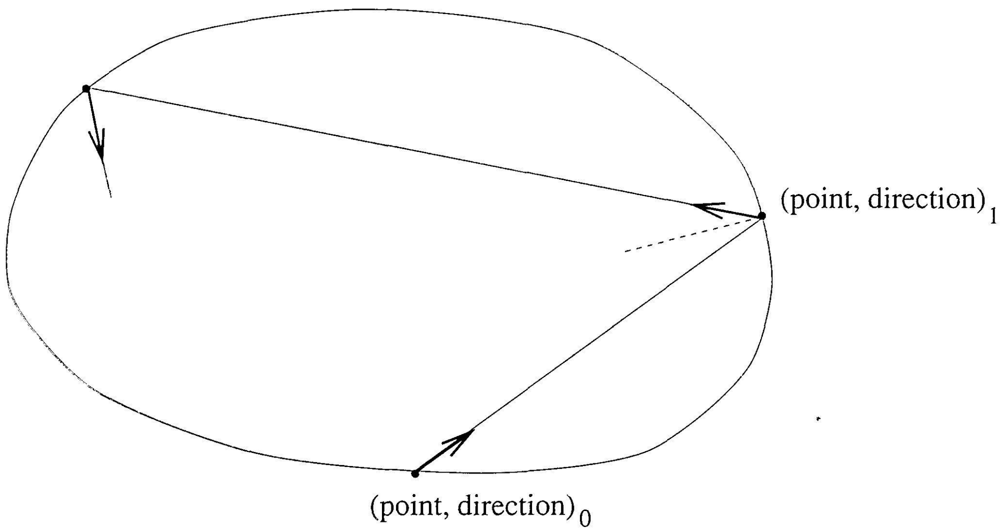

FIGURE 1
A strictly convex billiard in the plane

EXAMPLE 3. A particularly interesting class of monotone twist maps comes from planar convex billiards. The investigation of such systems goes back to Birkhoff who introduced them as a model case for nonlinear dynamical systems; for a modern survey see [Ta]. Given a strictly convex domain Ω

in the Euclidean plane with smooth boundary $\partial\Omega$, we play the following game. Let a mass point move freely inside Ω, starting at some initial point on the boundary with some initial direction pointing into Ω; when the "billiard ball" hits the boundary, it gets reflected according to the rule "angle of incidence = angle of reflection". The billiard map associates to a pair (point on the boundary, direction), respectively (s, φ) = (arclength parameter divided by total length, angle with the tangent), the corresponding data when the points hits the boundary again; see Figure 1. This map, which is defined on $\mathbf{S}^1 \times (0, \pi)$, is not a monotone twist map. However, elementary geometry shows that it preserves the 2-form $\sin\varphi\, d\varphi \wedge ds = d(-\cos\varphi) \wedge ds$. Hence the billiard map preserves the standard area form $dx \wedge dy$ in the new coordinates $(x, y) = (s, -\cos\varphi) \in \mathbf{S}^1 \times (-1, 1)$. Moreover, if you increase the angle with the positive tangent to $\partial\Omega$ for the initial direction, the point where you hit $\partial\Omega$ again will move around $\partial\Omega$ in positive direction. This means that $\partial x_1/\partial y_0 > 0$, so the billiard map satisfies the monotone twist condition.

EXAMPLE 4. Consider a particle moving in a potential on the line. According to Newton's Second Law, the motion of the particle is determined by the differential equation $\ddot{x}(t) = V'(x(t))$. This can be written as a Hamiltonian system $\dot{x} = \partial H(x, y)/\partial y$, $\dot{y} = -\partial H(x, y)/\partial x$ with the Hamiltonian $H(x, y) = y^2/2 - V(x)$. For small enough $t > 0$, we have

$$\frac{\partial x(t; x(0), y(0))}{\partial y(0)} = \frac{\partial}{\partial y(0)} \int_0^t \dot{x}(\tau; x(0), y(0))\, d\tau = \int_0^t \frac{\partial y(\tau; x(0), y(0))}{\partial y(0)}\, d\tau > 0\,.$$

Therefore the time-t-map φ_H^t is a monotone twist map provided t is small.

A particular case is that of a mathematical pendulum where x is the angle to the vertical and $V'(x) = -\sin 2\pi x$. The phase portrait in $\mathbf{S}^1 \times \mathbf{R}$ shows two types of invariant curves: contractible ones around the stable equilibrium ("librational" circles) and homotopically nontrivial ones above and below the separatrices ("rotational" curves).

A classical theorem of Birkhoff (see [Bi1, §44] and [Bi2, §3]) says that an embedded, closed, homotopically nontrivial curve, which is invariant under a monotone twist map, must be the graph of a Lipschitz continuous function on $\mathbf{S}^1$. This is a strong consequence of the fact that the map under consideration is a monotone twist map. The assertion does not follow, of course, if one drops the monotone twist condition (just take the identity mapping); nor is it valid without the area-preserving property (see [LCa, Prop. 15.3] for a counterexample). Finally, we have seen in Example 4 that a monotone twist map can perfectly possess embedded invariant circles which are not graphs, but they are homotopically trivial.

For strictly convex billiards, Birkhoff's Theorem states that invariant curves of the billiard map correspond to so-called caustics; these are continuous curves inside Ω with the property that a billiard trajectory, which is tangent to the caustic, stays tangent to it after one reflection.

Birkhoff's Theorem can also be used to derive non-existence results for invariant curves. It implies, for instance, that for convex billiards which are not strictly convex, there are no caustics at all [Ma1]. Furthermore, Birkhoff's Theorem provides a useful criterion for the non-existence of invariant curves for the standard map. This criterion, together with numerical calculations, pushed the parameter bound, above which the standard map possesses no invariant curves anymore, down to 63/64 [MP].

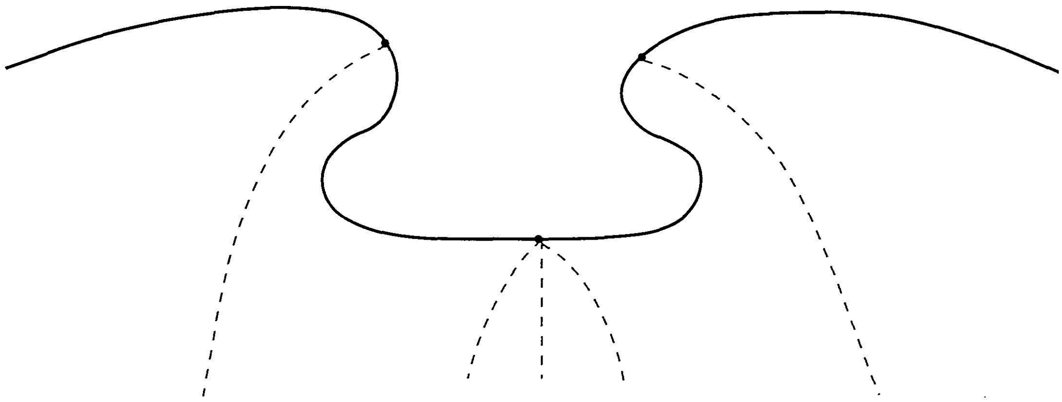

FIGURE 2
An invariant curve which is not a graph

There are several proofs of different versions of Birkhoff's Theorem [Fa, He, KH, Ma2, Ma3]. All of them are based on Birkhoff's ideas and use topological arguments. Their common idea is to consider two kinds of points on an invariant closed curve: those accessible by rays that originate from the lower end of the cylinder and are tilted to the right, and those accessible by rays tilted to the left; see Figure 2. It is shown that these two classes coincide, and hence every point on the invariant curve is accessible by a vertical ray. This latter fact, however, obvious as it may seem, is not trivial.

The aim of this note is to introduce a different approach to proving Birkhoff's Theorem, involving a new iteration argument. Assume that ϕ possesses an invariant curve Γ that is not a graph but folded over the x-axis. Then, due to the fact that ϕ is an area-preserving twist map, one application of ϕ presses some more area into that fold. Iterating this procedure, we see that the folds will enclose larger and larger domains. Their areas, however, stay bounded since Γ is an invariant curve on the cylinder. Therefore those additional areas must tend to zero. But this can only happen if Γ has a point of self-intersection, which contradicts its embeddedness.

I would like to thank Patrice Le Calvez for drawing my attention to the fact that Birkhoff's Theorem is not true without the area-preserving assumption, as well as Martin Beibel (from the Institute for Mathematical Stochastics, University of Freiburg) for reading and commenting on a preliminary version. This proof was presented in one of those evening sessions during the Dynamical Systems meeting in Oberwolfach (1997), and I thank everyone in the audience for attending.

2. Birkhoff's Theorem

We consider a C^1-diffeomorphism $\phi\colon \mathbf{S}^1 \times \mathbf{R} \to \mathbf{S}^1 \times \mathbf{R}$ of the two-dimensional cylinder; for the sake of simplicity, we keep the same notation for a lift of ϕ to $\mathbf{R}^2$ with coordinates x, y.

Definition. We say that ϕ is a *monotone twist mapping* if the following three conditions hold:

- $\phi^*(dx \wedge dy) = dx \wedge dy$, i.e. ϕ preserves area and orientation.
- $\pi_y \circ \phi(x, y) \to \pm\infty$ as $y \to \pm\infty$, i.e. ϕ preserves the ends of the cylinder.
- $|\partial(\pi_x \circ \phi)/\partial y| \geq \delta > 0$, i.e. ϕ satisfies a uniform monotone twist condition.

According to the sign of $\partial(\pi_x \circ \phi)/\partial y$, we call ϕ a *positive*, respectively *negative*, monotone twist mapping.

The uniformity of the twist condition has the following geometric interpretation ("cone condition"). Let ϕ be a positive monotone twist map, and denote by v_x the vertical $\{x\} \times \mathbf{R}$. Then the image $\phi(v_x)$ crosses the vertical through $\phi(x, y)$ in positive direction and stays outside a cone around it with centre $\phi(x, y)$, whose angle depends only on the twist constant δ; see Figure 3.

Note that if ϕ is a positive monotone twist mapping then its inverse ϕ^{-1} is a negative monotone twist mapping.

For the statement of the theorem, recall that a closed continuous curve is embedded if it is homeomorphic to $\mathbf{S}^1$; in particular, it cannot have a point of self-intersection.