2. Some definitions

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First, we recall what Gromov's genericity is.

Definition (Champetier). Consider two integers $k \geq 2$, $l \geq 1$, a set X of k generators and a property P of group presentations with X as generating system and with l relations. For integers $n_1, \dots, n_l \geq 1$, let $Pr(X, n_1, \dots, n_l)$ denote the finite set of presentations $\langle X \mid r_1, \dots, r_l \rangle$ where r_i is a cyclically reduced relation in the generators of X which is of length $|r_i| = n_i$ $(1 \leq i \leq l)$.

Then P is said to be generic in the sense of Gromov if the ratio

$$\frac{\#\{\langle X \mid R \rangle \in Pr(X, n_1, \dots, n_l) \mid \langle X \mid R \rangle \text{ satisfies } P\}}{\# Pr(X, n_1, \dots, n_l)}$$

tends to 1 when $\min\limits_{i=1,\dots,l} n_i \to +\infty$.

For example, being a hyperbolic group is a generic property. This was proved independently by Champetier [5] and Ol′shanskii [13].

One tool we need is small cancellation theory. Let $\langle X \mid R \rangle$ be a presentation of a group Γ. Denote by R^* the set of cyclic conjugates of elements of R and of their inverses.

Definition 2.1. Let $\Gamma = \langle X \mid R \rangle$ be a finitely presented group. A *piece* is a prefix u common to at least two distincts elements in R^* (by prefix, we mean every non empty initial part of a word; in particular a word is a particular prefix for itself).

Fix $\lambda \in]0, 1[$. The presentation $\langle X \mid R \rangle$ satisfies the *small cancellation condition* $C'(\lambda)$ if the following inequality holds: $|u| < \lambda |r|$ for every $r \in R^*$ and for every prefix u of r which is a piece.

Definition 2.2. A group $\Gamma = \langle X \mid R \rangle$ satisfies a *Dehn algorithm* if, for every non trivial reduced word $\omega \in \mathbf{F}_X$ representing 1 in Γ, there exists a prefix u of some word $r \in R^*$ such that u is a subword of ω and $|u| > \frac{1}{2}|r|$.

It is known that groups satisfying the small cancellation condition $C'(1/6)$ also admit a Dehn algorithm (see Theorem 4.4, Chapter V in [11] or Theorem 25 in [14]). On the other hand Gromov proves that groups with a Dehn algorithm are hyperbolic (see [8, Theorem 2.3.D]).

In Proposition 4.1 below, $C'(1/6)$ is one of the conditions which imply that, for some fixed $x_0 \in X$, $X - \{x_0\}$ generates a free subgroup in Γ.

Let $\langle X \mid R \rangle$ be a presentation with k generators and l relations $r_1, \ldots, r_l$. G. Arzhantseva and A. Ol′shanskii proved, in [1], that for any fixed $\lambda > 0$,

$$\lim_{d \to +\infty} \frac{\#\{\langle X \mid R \rangle \text{ with } C'(\lambda) \mid \sum_{i=1}^{l} |r_i| = d, \ r_i \text{ cyclically reduced}\}}{\#\{\langle X \mid R \rangle \mid \sum_{i=1}^{l} |r_i| = d, \ r_i \text{ cyclically reduced}\}} = 1 .$$

Unfortunately, even with this result, it is not known if the small cancellation hypothesis is generic, so we need another hypothesis which is generic. Let us recall the definition of Van Kampen diagrams.

DEFINITION 2.3. Let $\omega \in \mathbf{F}_X$ represent the identity in $\Gamma = \langle X \mid R \rangle$. Then Δ is a *Van Kampen diagram* of ω if Δ is a planar 2-complex for which the 1-skeleton is a graph, each edge of it being labelled by a element of X or X^{-1} such that when we read the labelling of every 2-cell of the complex, we get a word in R^*, and such that the labelling of the border of the complex Δ is the word ω.

For more details about Van Kampen diagrams, see [14], [3] or [11]. We denote by $I(\Delta)$ (resp. $E(\Delta)$ and $\#(\Delta)$) the number of internal edges of Δ (resp. the number of external edges of Δ and the total number of edges of Δ).

DEFINITION 2.4. The *combinatorial area* of a Van Kampen diagram Δ is the number of its 2-cells. We say that Δ is a reduced diagram of ω if it has the minimal combinatorial area among all diagrams representing ω.

For every $\omega \in \mathbf{F}_X$ representing the identity in $\Gamma = \langle X \mid R \rangle$, the existence of such a reduced diagram of ω is proved in [3].

DEFINITION 2.5. For $0 < \theta < 1$, a finite presentation $\langle X \mid R \rangle$ is said to satisfy the *θ-condition*, if for every reduced diagram Δ associated with a reduced word ω in $\mathbf{F}_X$ representing the identity in $\langle X \mid R \rangle$, we have $I(\Delta) < \theta \#(\Delta)$.

In [13], Ol′shanskii showed that for every fixed $\theta > 0$, the property of satisfying a θ-condition is generic.

To prove that result, he needed to introduce the following definition.

DEFINITION 2.6. A reduced diagram is *simple* if every edge is contained in the boundary of a 2-cell of the diagram.

It is clear that every reduced diagram of ω is a disjoint union of simple ones linked by bridges, where a bridge is a finite path of edges which are not in the boundary of a 2-cell, and, because the word ω in $\mathbf{F}_X$ is reduced, each bridge links two simple diagrams. In figure 1 the diagram contains three simple diagrams (D1, D2, D3) and two bridges (B1, B2).

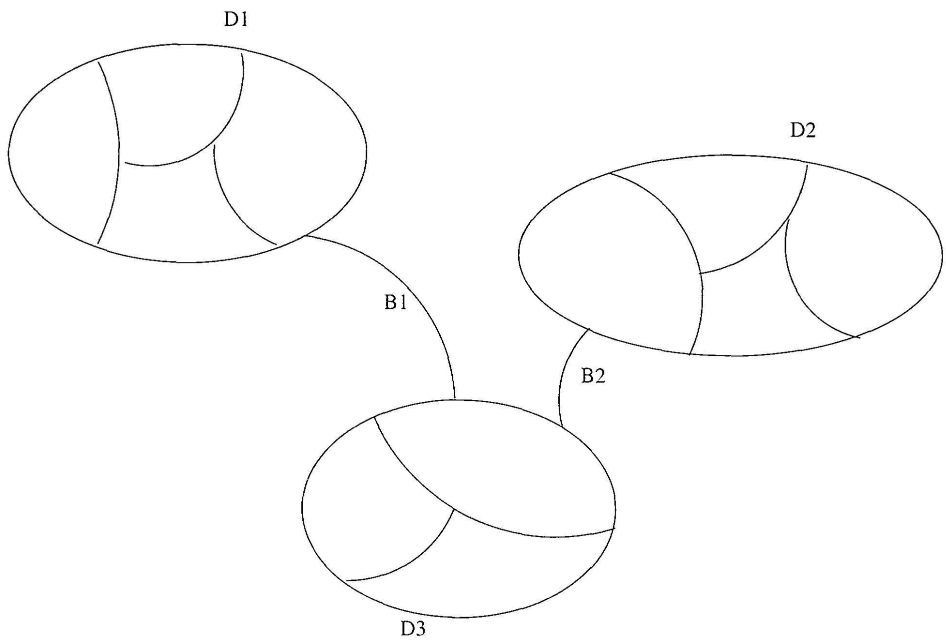

FIGURE 1
A non simple diagram

Let X be a set of generators and $y \in X$. For every reduced word $r \in \mathbf{F}_X$, we denote by $n_y(r)$ the number of occurences of y and y^{-1} in r. For example $n_y(yx^3y^{-2}xy^3) = 6$.

DEFINITION 2.7. Let $\mathbf{F}_X$ be the free group on X with $\#X = k$. For a fixed ϵ with $0 < \epsilon < 1/k$ and $y \in X$, a non trivial reduced word $r \in \mathbf{F}_X$ is (ϵ, y)-balanced if

$$\frac{n_y(r)}{|r|} \geq \epsilon .$$

A presentation $\Gamma = \langle X \mid R \rangle$ is (ϵ, y)-balanced, if every $r \in R$ is *(ϵ, y)-balanced.*