2. PRELIMINAIRES AND NOTATION

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of G in $GL_d(\mathbf{Q})$ to test strong modularity of L. In the next section we derive some methods for explicitly constructing elements of $N_{GL_d(\mathbf{Q})}(G)$.

Every finite subgroup of $GL_d(\mathbf{Q})$ is a subgroup of the automorphism group of an integral lattice. In particular the maximal finite subgroups of $GL_d(\mathbf{Q})$ are automorphism groups of distinguished lattices. A subgroup of $GL_d(\mathbf{Q})$ is called rational irreducible if it does not preserve a proper subspace $\neq \{0\}$ of $\mathbf{Q}^d$. The rational irreducible maximal finite, abbreviated to *r.i.m.f.*, subgroups of $GL_d(\mathbf{Q})$ are classified for $d < 32$ (cf. [PlN 95], [NeP 95], [Neb 95], [Neb 96], [Neb 96a]). Their invariant lattices provide many examples of strongly modular lattices. The following theorem is proved by applying the methods derived in Section 4.

THEOREM. *In dimension $d < 32$, all even lattices $L \subseteq \mathbf{R}^d$ that are preserved by a r.i.m.f. group and satisfy $L^{\#}/L \cong (\mathbf{Z}/l\mathbf{Z})^{d/2}$ for some $l \in \mathbf{N}$ are strongly modular, except for the lattices of the r.i.m.f. group* $[\pm \mathrm{Alt}_6 .2^2]_{16}$ *in $GL_{16}(\mathbf{Q})$ (cf.* [NeP 95]*).*

2. Preliminaries and Notation

The main strategy in this paper is the application of the following *normaliser principle.*

Let G be a group acting on a set S, H a subgroup of the group of transformations of S. Then the normaliser of G in H acts on the set of G-orbits.

In our situation $G = \mathrm{Aut}(L)$ is the automorphism group of an integral lattice L in the Euclidean space $\mathbf{R}L \cong \mathbf{R}^d$. By writing the action of G on $\mathbf{R}L$ with respect to a $\mathbf{Z}$-basis $(b_1, \ldots, b_d)$ of L, G becomes a finite subgroup of $GL_d(\mathbf{Z})$. Then $G = \mathrm{Aut}(F) = \{g \in GL_d(\mathbf{Z}) \mid gFg^{tr} = F\}$ where F is the Gram matrix $F = \left((b_i, b_j)\right)_{i,j=1}^{d}$ of L.

For the rest of this article let $H = GL_d(\mathbf{Q})$, $G \leq H$, be a finite subgroup of H, and let $N := N_H(G)$ be its normaliser. We also assume that G contains the negative unit matrix, $-I_d \in G$.

We apply the normaliser principle to the following three situations.

(i) $S = \{L \subseteq \mathbf{Q}^d \mid L = \sum_{i=1}^{d} \mathbf{Z}b_i$ for a basis $(b_1, \ldots, b_d)$ of $\mathbf{Q}^d\}$, the set of $\mathbf{Z}$-lattices of rank d in $\mathbf{Q}^d$, and the action of H on S is right multiplication: $S \times H \to S$, $(L, h) \mapsto Lh := \{lh \mid l \in L\}$. Then the set of G-fixed points is

$$\mathcal{Z}(G) := \{L \in S \mid Lg = L \text{ for all } g \in G\},$$

the set of G-invariant lattices.

(ii) $S = \{F \in M_d(\mathbf{Q}) \mid F = F^{tr},\ F$ positive definite$\}$, the set of positive definite symmetric matrices, where x^{tr} denotes the transposed matrix of $x \in M_d(\mathbf{Q})$ and the action of H on S is $S \times H \to S$, $(F,h) \mapsto hFh^{tr}$. Then the set of G-fixed points is

$$\mathcal{F}_{>0}(G) := \{F \in S \mid gFg^{tr} = F \text{ for all } g \in G\}.$$

Note that $(\mathbf{R}_{>0})\mathcal{F}_{>0}(G)$ is the set of G-invariant Euclidean scalar products on $\mathbf{R}^d$. G is called *uniform*, if there is essentially one G-invariant Euclidean structure on $\mathbf{R}^d$, that is if $\mathcal{F}_{>0}(G) = \{aF \mid 0 < a \in \mathbf{Q}\}$ for some $F \in M_d(\mathbf{Q})$.

(iii) $S = M_d(\mathbf{Q})$, and the action of H is conjugation: $S \times H \to S$, $(c,h) \mapsto h^{-1}ch$. Then the set of G-fixed points is the *commuting algebra* of G

$$C_{M_d(\mathbf{Q})}(G) := \{c \in M_d(\mathbf{Q}) \mid cg = gc \text{ for all } g \in G\}.$$

The following two remarks follow immediately from the normaliser principle.

REMARK 1. *Assume that G is uniform and let $F \in \mathcal{F}_{>0}(G)$. Then for each $n \in N$, the matrix nFn^{tr} is also G-invariant and therefore $nFn^{tr} = (\det(n))^{2/d}F$. Hence n induces a similarity of F.*

REMARK 2. *For $n \in N$ and $L \in \mathcal{Z}(G)$, the lattice $Ln \in \mathcal{Z}(G)$ is also G-invariant.*

3. SIMILARITIES NORMALISE

In this section we show that if G is the automorphism group of a (strongly modular) lattice L then the similarities between L and $L' \in \pi(L)$ are elements of N.

PROPOSITION 3. *Let $G = \mathrm{Aut}(F) \leq GL_d(\mathbf{Z})$ be the full automorphism group of a lattice L. Assume that L is an integral lattice. Let $L' \in \pi(L)$ and $n \in GL_d(\mathbf{Q})$ which induces a similarity from L' to L, i.e. $L'n = L$ and $nFn^{tr} = aF$, $(a \in \mathbf{N})$. Then $a^{-1}n^2 \in G$ and $n \in N$.*

Proof. The matrix $a^{-1}n^2$ is clearly orthogonal with respect to F. Therefore to prove that $a^{-1}n^2 \in G$ we only have to show that $La^{-1}n^2 = L$. Now $L' = Ln^{-1}$, hence its dual lattice is

$$(L')^{\#} = \{v \in \mathbf{Q}^d \mid vF(ln^{-1})^{tr} \in \mathbf{Z} \quad \text{for all} \quad l \in L\}.$$