

## 2. PRELIMINAIRES AND NOTATION

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of  $G$  in  $GL_d(\mathbf{Q})$  to test strong modularity of  $L$ . In the next section we derive some methods for explicitly constructing elements of  $N_{GL_d(\mathbf{Q})}(G)$ .

Every finite subgroup of  $GL_d(\mathbf{Q})$  is a subgroup of the automorphism group of an integral lattice. In particular the maximal finite subgroups of  $GL_d(\mathbf{Q})$  are automorphism groups of distinguished lattices. A subgroup of  $GL_d(\mathbf{Q})$  is called rational irreducible if it does not preserve a proper subspace  $\neq \{0\}$  of  $\mathbf{Q}^d$ . The rational irreducible maximal finite, abbreviated to *r.i.m.f.*, subgroups of  $GL_d(\mathbf{Q})$  are classified for  $d < 32$  (cf. [PIN 95], [NeP 95], [Neb 95], [Neb 96], [Neb 96a]). Their invariant lattices provide many examples of strongly modular lattices. The following theorem is proved by applying the methods derived in Section 4.

**THEOREM.** *In dimension  $d < 32$ , all even lattices  $L \subseteq \mathbf{R}^d$  that are preserved by a r.i.m.f. group and satisfy  $L^\# / L \cong (\mathbf{Z} / l\mathbf{Z})^{d/2}$  for some  $l \in \mathbf{N}$  are strongly modular, except for the lattices of the r.i.m.f. group  $[\pm \text{Alt}_6 . 2^2]_{16}$  in  $GL_{16}(\mathbf{Q})$  (cf. [NeP 95]).*

## 2. PRELIMINARIES AND NOTATION

The main strategy in this paper is the application of the following *normaliser principle*.

Let  $G$  be a group acting on a set  $S$ ,  $H$  a subgroup of the group of transformations of  $S$ . Then the normaliser of  $G$  in  $H$  acts on the set of  $G$ -orbits.

In our situation  $G = \text{Aut}(L)$  is the automorphism group of an integral lattice  $L$  in the Euclidean space  $\mathbf{R}L \cong \mathbf{R}^d$ . By writing the action of  $G$  on  $\mathbf{R}L$  with respect to a  $\mathbf{Z}$ -basis  $(b_1, \dots, b_d)$  of  $L$ ,  $G$  becomes a finite subgroup of  $GL_d(\mathbf{Z})$ . Then  $G = \text{Aut}(F) = \{g \in GL_d(\mathbf{Z}) \mid gFg^{tr} = F\}$  where  $F$  is the Gram matrix  $F = ((b_i, b_j))_{i,j=1}^d$  of  $L$ .

For the rest of this article let  $H = GL_d(\mathbf{Q})$ ,  $G \leq H$ , be a finite subgroup of  $H$ , and let  $N := N_H(G)$  be its normaliser. We also assume that  $G$  contains the negative unit matrix,  $-I_d \in G$ .

We apply the normaliser principle to the following three situations.

- (i)  $S = \{L \subseteq \mathbf{Q}^d \mid L = \sum_{i=1}^d \mathbf{Z}b_i \text{ for a basis } (b_1, \dots, b_d) \text{ of } \mathbf{Q}^d\}$ , the set of  $\mathbf{Z}$ -lattices of rank  $d$  in  $\mathbf{Q}^d$ , and the action of  $H$  on  $S$  is right multiplication:  $S \times H \rightarrow S$ ,  $(L, h) \mapsto Lh := \{lh \mid l \in L\}$ . Then the set of  $G$ -fixed points is

$$\mathcal{Z}(G) := \{L \in S \mid Lg = L \text{ for all } g \in G\},$$

the set of  $G$ -invariant lattices.

- (ii)  $S = \{F \in M_d(\mathbf{Q}) \mid F = F^{tr}, F \text{ positive definite}\}$ , the set of positive definite symmetric matrices, where  $x^{tr}$  denotes the transposed matrix of  $x \in M_d(\mathbf{Q})$  and the action of  $H$  on  $S$  is  $S \times H \rightarrow S$ ,  $(F, h) \mapsto hFh^{tr}$ . Then the set of  $G$ -fixed points is

$$\mathcal{F}_{>0}(G) := \{F \in S \mid gFg^{tr} = F \text{ for all } g \in G\}.$$

Note that  $(\mathbf{R}_{>0})\mathcal{F}_{>0}(G)$  is the set of  $G$ -invariant Euclidean scalar products on  $\mathbf{R}^d$ .  $G$  is called *uniform*, if there is essentially one  $G$ -invariant Euclidean structure on  $\mathbf{R}^d$ , that is if  $\mathcal{F}_{>0}(G) = \{aF \mid 0 < a \in \mathbf{Q}\}$  for some  $F \in M_d(\mathbf{Q})$ .

- (iii)  $S = M_d(\mathbf{Q})$ , and the action of  $H$  is conjugation:  $S \times H \rightarrow S$ ,  $(c, h) \mapsto h^{-1}ch$ . Then the set of  $G$ -fixed points is the *commuting algebra* of  $G$

$$C_{M_d(\mathbf{Q})}(G) := \{c \in M_d(\mathbf{Q}) \mid cg = gc \text{ for all } g \in G\}.$$

The following two remarks follow immediately from the normaliser principle.

REMARK 1. Assume that  $G$  is uniform and let  $F \in \mathcal{F}_{>0}(G)$ . Then for each  $n \in N$ , the matrix  $nFn^{tr}$  is also  $G$ -invariant and therefore  $nFn^{tr} = (\det(n))^{2/d}F$ . Hence  $n$  induces a similarity of  $F$ .

REMARK 2. For  $n \in N$  and  $L \in \mathcal{Z}(G)$ , the lattice  $Ln \in \mathcal{Z}(G)$  is also  $G$ -invariant.

### 3. SIMILARITIES NORMALISE

In this section we show that if  $G$  is the automorphism group of a (strongly modular) lattice  $L$  then the similarities between  $L$  and  $L' \in \pi(L)$  are elements of  $N$ .

PROPOSITION 3. Let  $G = \text{Aut}(F) \leq GL_d(\mathbf{Z})$  be the full automorphism group of a lattice  $L$ . Assume that  $L$  is an integral lattice. Let  $L' \in \pi(L)$  and  $n \in GL_d(\mathbf{Q})$  which induces a similarity from  $L'$  to  $L$ , i.e.  $L'n = L$  and  $nFn^{tr} = aF$ , ( $a \in \mathbf{N}$ ). Then  $a^{-1}n^2 \in G$  and  $n \in N$ .

*Proof.* The matrix  $a^{-1}n^2$  is clearly orthogonal with respect to  $F$ . Therefore to prove that  $a^{-1}n^2 \in G$  we only have to show that  $La^{-1}n^2 = L$ . Now  $L' = Ln^{-1}$ , hence its dual lattice is

$$(L')^\# = \{v \in \mathbf{Q}^d \mid vF(ln^{-1})^{tr} \in \mathbf{Z} \text{ for all } l \in L\}.$$