1. Standard involutions on lattices

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1. Standard involutions on lattices

Throughout the paper Λ denotes an even lattice on euclidean n-space and Λ^* its dual lattice on the same space. So $x \cdot x \in 2\mathbf{Z}$ holds for all $x \in \Lambda$, and $\Lambda \subset \Lambda^*$. The group $D = \Lambda^*/\Lambda$ carries the $\mathbf{Q}/\mathbf{Z}$-valued regular quadratic form $\varphi(\overline{v}) = \frac{1}{2} v \cdot v + \mathbf{Z}$, where $\overline{v} = v + \Lambda$ for $v \in \Lambda^*$. We choose an integer $\ell > 0$ such that also $\sqrt{\ell}\Lambda^*$ is even — then Λ is said to be of *level* ℓ. It follows that $\ell\Lambda^* \subset \Lambda$, and φ takes its values in $\frac{1}{\ell}\mathbf{Z}/\mathbf{Z}$. Note that $\det \Lambda = \# D$ divides $\ell^n = (\det \Lambda)(\det \sqrt{\ell}\Lambda^*)$. We are especially interested in the situation when both factors are equal, i.e. $\det \Lambda = \ell^k$ for $n = 2k$.

Let $\ell = mm'$ with coprime integers $m, m' > 0$ (notation: $m \| \ell$), and let $D(m)$ be the m-torsion subgroup of D. Clearly, $D = D(m) \oplus D(m')$, an orthogonal decomposition with respect to the discriminant form. It is a standard procedure to associate with Λ and m that lattice between Λ and Λ^* (suitably rescaled) whose image in D is $D(m)$. We summarize its main properties.

Proposition 1. *Let Λ be of level $\ell = mm'$ as above. Then also*

$$\Lambda_m = \sqrt{m}\left(\Lambda^* \cap \frac{1}{m}\Lambda\right)$$

is an even lattice of level ℓ, satisfying

$$\text{(i)} \quad \sqrt{\ell}\Lambda_m^* = \Lambda_{m'} \quad \text{and}$$
$$\text{(ii)} \quad (\Lambda_m)_m = \Lambda, \qquad (\Lambda_m)_{m'} = \Lambda_\ell .$$

If $\det \Lambda = \ell^k$ *holds for* $k = \frac{n}{2}$, *then also* $\det \Lambda_m = \ell^k$.

Proof. First Λ_m is even because $\sqrt{m}\Lambda_m \subset \Lambda$ and $\sqrt{m'}\Lambda_m \subset \sqrt{\ell}\Lambda^*$. Property (i) follows from $D(m)^\perp = D(m')$ and implies that Λ_m is of level ℓ. Also (ii) easily follows from (i). Finally $\# D = \ell^k$ implies $\# D(m) = m^k$, and so $\det \Lambda_m = \ell^k m^{n-2k}$. □

Suppose that $\det \Lambda = \ell^{n/2}$. If, moreover, Λ is isometric to Λ_m for all $m \| \ell$, then Λ is called *strongly modular*. So the classes of such lattices are the common fixed points under the group of involutions defined by Proposition 1 on the set of all classes of level ℓ.

We take a quick look at dimension $n = 2$, assuming ℓ to be squarefree, therefore $\ell \equiv 3 \pmod 4$, and using composition theory (cf. [Ca], Ch.14). An isometry class of lattices of level ℓ then corresponds to a set $\{C, C^{-1}\}$ where C is an ideal class of the ring of integers in $\mathbf{Q}(\sqrt{-\ell})$. Our group of involutions is given by the ideal classes of exponent 2 acting by multiplication. It turns out that strong modularity cannot occur unless ℓ is a prime or a product of two primes which are quadratic residues of each other. In the first case there are no nontrivial involutions, while in the second case there is precisely one fixed point $\{C, C^{-1}\}$, given by the elements of order 4 in the ideal class group (whose 2-primary part in this case is cyclic of order at least 4).

Before going on, we recall some facts on Gaussian sums; for the proofs see [Sc], Ch. 5. These invariants of quadratic forms are also most natural in connection with modular forms (see next section). We put $e(z) = e^{2\pi i z}$.

PROPOSITION 2. *Let Λ, m and $D(m)$ be as in Proposition 1. Then*

$$g_m(\Lambda) = \left(\#D(m)\right)^{-\frac{1}{2}} \sum_{\overline{v} \in D(m)} e\left(\frac{1}{2} v \cdot v\right)$$

is an eighth root of unity depending only on the isometry class of the rational quadratic space $\Lambda \otimes \mathbf{Q}$ (and on m). Furthermore,

$$g_m(\Lambda) g_{m'}(\Lambda) = g_\ell(\Lambda) = e\left(\frac{n}{8}\right).$$

When m is a prime p and $k = \dim_{\mathbf{F}_p} D(p)$,

$$g_p(\Lambda) = \begin{cases} \pm i^k & \text{if } p \equiv 3 \pmod 4 \\ \pm 1 & \text{otherwise}. \end{cases}$$

The ambiguity of signs above corresponds to the two possibilities for the isometry class of a k-dimensional regular quadratic space over $\mathbf{F}_p$. When k is even, $g_p(\Lambda) = 1$ holds if and only if $D(p)$ is hyperbolic. In general, the *genus* of Λ may be defined as the isometry class of D. In particular, for squarefree ℓ it is determined by $\det \Lambda$ and all $g_p(\Lambda)$, where p (prime) divides ℓ. The rest of this section deals with the special case

$$\ell = pq,\ p \neq q \text{ primes},\ n = 2k,\ k \text{ even},\ \det \Lambda = \ell^k.$$

Here all Λ having $g_p(\Lambda) = \varepsilon$, $g_q(\Lambda) = \delta$ form one genus, denoted by $G_n(p^\varepsilon q^\delta)$, and there are two such genera subject to the conditions $\varepsilon, \delta \in \{-1, +1\}$, $\varepsilon\delta = i^k$. As a whole, each genus is invariant under the standard involutions.

EXAMPLE 1. Let $p \equiv 3 \pmod 4$ and L be any 2-dimensional even lattice of determinant p. Then it is easy to see that the orthogonal sum $L \oplus \sqrt{q}L$ is a strongly modular lattice in $G_4(p^{\varepsilon}q^{-\varepsilon})$ for the Legendre symbol $\varepsilon = \left(\frac{-q}{p}\right)$. So we have such lattices in $G_4(2^-3^+)$, $G_4(2^+7^-)$, $G_4(3^+5^-), \ldots$. For arbitrary primes p and q we obtain, by the same construction, a strongly modular lattice in $G_8(p^+q^+)$ from a 4-dimensional one of level p (which always exists, cf. [Qu]).

EXAMPLE 2. Here we use some information on (proper) class numbers from [Vi], p. 153. First also $G_4(2^+3^-)$, $G_4(2^-5^+)$ and $G_4(2^+5^-)$ contain strongly modular lattices because they have class number 1. On the other hand, let Λ be the sublattice of the D_4 root lattice formed by all $x = (x_1, \ldots, x_4)$ such that $x_1 + 2x_2 + 3x_3 \equiv 0 \pmod 7$. It is easy to see that Λ belongs to $G_4(2^-7^+)$ and that $\min \Lambda = 4$. (As usual, $\min \Lambda$ denotes the minimum "norm" $x \cdot x$ for $x \in \Lambda$, $x \neq 0$.) Since $(0,0,0,2)$ sits in $2\Lambda^* \cap \Lambda$, we have $\min \Lambda_2 = 2$, and so Λ_2 is not isometric to Λ. Since $G_4(2^-7^+)$ has class number 2, no strongly modular lattice exists in this genus. Similarly, also $G_4(3^-5^+)$ contains no such lattice. In this case the two classes are represented by $L \oplus L$, where L is the binary lattice with Gram matrix $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 8 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 4 & 1 \\ 1 & 4 \end{smallmatrix}\right)$.

EXAMPLE 3. Again, let $p \equiv 3 \pmod 4$, q odd, and $K = \mathbf{Q}(\alpha, \beta)$, where $\alpha^2 = -p$ and $\beta^2 = (-1)^{(q-1)/2}q$. Let $\mathcal{O}_K$ be the ring of integers of K; its different is generated by $\lambda = \alpha\beta$. Given a totally positive hermitian space (V, h) of dimension r over K, we also consider it as an inner product space of dimension $n = 4r$ over $\mathbf{Q}$ for

$$x \cdot y = Tr_{K/\mathbf{Q}}\big(h(x,y)\big).$$

Then, if Λ is an $\mathcal{O}_K$-lattice on V with hermitian dual lattice Λ^h, its euclidean dual is $\Lambda^* = \lambda^{-1}\Lambda^h$. Therefore, a unimodular hermitian lattice, when considered euclidean, belongs to the r-fold sum of the 4-dimensional genus occurring in Example 1 and is strongly modular (before rescaling, Λ_p and Λ_q are given by $\alpha^{-1}\Lambda$ and $\beta^{-1}\Lambda$, respectively). Furthermore, inspection of the traces of totally-positive elements in real quadratic fields shows that we always have $\min \Lambda \geq 4$, moreover, $\min \Lambda \geq 6$ if there is no $x \in \Lambda$ satisfying $h(x,x) = 1$ (in particular, if $r \geq 2$ and (Λ, h) is indecomposable), and even $\min \Lambda \geq 8$ if this condition holds for $q \neq 5$.

EXAMPLE 4. Similarly, $K = \mathbf{Q}\left(e\left(\frac{1}{8}\right), \sqrt{\pm q}\right)$ may be used to obtain strongly modular lattices Λ in $G_{8r}(2^+q^+)$ from rank r unimodular hermitian lattices over $\mathcal{O}_K$, setting now

$$x \cdot y = \frac{1}{2}\, Tr_{K/\mathbf{Q}}\big(h(x,y)/(2-\sqrt{2})\big)\,.$$

Again, we always have $\min \Lambda \geq 4$. E.g., $\mathcal{O}_K$ itself gives the tensor product of D_4 and the binary lattice with Gram matrix $\left(\begin{smallmatrix} 2 & 1 \\ 1 & (q+1)/2 \end{smallmatrix}\right)$.

2. ATKIN-LEHNER ACTION ON THETA FUNCTIONS

The subject treated in this section is not new, but appears to be difficult to cite from the literature (in the form we need it). For convenience, I give a rather detailed account, starting from a classical formula (due to Jacobi and others). Let Λ be an even lattice. The theta function of a coset $\overline{v} = v + \Lambda$ in Λ^* (and, in particular, Θ_Λ for $v = 0$) is that function defined on the upper half-plane by

$$\Theta_{\overline{v}}(z) = \sum_{x \in \overline{v}} e\left(\frac{1}{2}(x \cdot x)\, z\right)\,.$$

Now let $n = 2k$ (k integral), and recall that $SL_2(\mathbf{R})$ acts on functions f on the upper half-plane by

$$(f\,|_k\, S)(z) = (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right), \qquad S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let S be in $SL_2(\mathbf{Z})$, with $c > 0$. For $u, v \in \Lambda^*$ define

$$\phi_S(u,v) = \sum_x e\big((ax \cdot x + 2x \cdot v + dv \cdot v)/2c\big)$$

where x runs through a system of representatives of those elements of $\Lambda^*/c\Lambda$ which reduce to $\overline{u}$ in $D = \Lambda^*/\Lambda$. Each summand clearly depends only on the class $x + c\Lambda$, and the whole sum depends only on $\overline{u}$ and $\overline{v}$. The latter statement is trivial for $\overline{u}$, while for $\overline{v}$ it is proved (using $1 = ad - bc$) by

$$\begin{aligned} \phi_S(u,v) &= \sum_x e\big(a(x+dv)\cdot(x+dv)/2c\big)\, e\big(-b(2x\cdot v + dv\cdot v)/2\big) \\ &= \phi_S(u+dv, 0)\, e\big(-b(2u\cdot v + dv \cdot v)/2\big)\,. \end{aligned} \tag{2.1}$$

So we may write $\phi_S(u,v) = \phi_S(\overline{u}, \overline{v})$. Then the formula we need is (see [Mi], p. 189)