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## AMENABILITY AND GROWTH OF ONE-RELATOR GROUPS

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ABSTRACT. An algorithm showing whether a group given by a one-relator presentation is amenable or not is constructed. Sufficient conditions for a one-relator group of exponential growth to have uniformly exponential growth are also given.

### 0. INTRODUCTION

A one-relator group is a group  $G$  which admits a presentation

$$(*) \quad G = \langle a_1, a_2, \dots, a_m : R(a_1, a_2, \dots, a_m) = 1 \rangle$$

with one defining relation.

The paper by G. Baumslag [B 1] is a comprehensive survey of results about one-relator groups. In particular this paper stresses the role of algorithmic problems in the theory of one-relator groups.

Recently the interest in functional-analytical and asymptotical properties of one-relator groups has increased. For instance, the entropy of one-relator groups was discussed in [GrLP], random walks and Markov operators on one-relator groups were investigated in [CV], [BCCH], [BC], and the K-functor of reduced  $C^*$ -algebras of one-relator groups was computed in [BBV]. Also the growth functions of the groups  $\Gamma_n = \langle t, a : tat^{-1} = a^n \rangle$ ,  $n \neq 0, \pm 1$ , and of some other one-relator groups were calculated in [CEG] and [EJ].

Recall that a discrete group  $G$  is amenable if there exists a finitely additive measure  $\mu: \mathcal{P}(G) = \{0, 1\}^G \rightarrow [0, 1]$  which is  $G$ -(left)-invariant ( $\mu(gE) = \mu(E)$  for all  $g \in G$  and  $E \subset G$ ) and such that, in addition,  $\mu(G) = 1$ . For our purpose it will be enough to know that a group containing a free subgroup of rank two is not amenable, and that, on the contrary, any solvable group is amenable ([G]).

As easily follows from the paper of Karrass and Solitar [KS], all amenable one-relator groups are in the following list:

- (\*\*) {
1.  $\langle a : a^n = 1 \rangle \cong \mathbf{Z}_n$ , cyclic groups of finite order  $n = 1, 2, \dots$ ;
  2.  $\langle a, b : b = 1 \rangle \cong \mathbf{Z}$ , the infinite cyclic group;
  3.  $\langle a, b : bab^{-1} = a^n \rangle$ ,  $n \neq 0$ .  
 This class splits into two subclasses:
    - 3<sub>a</sub>.  $n = +1$  :  $\langle a, b : bab^{-1} = a \rangle \cong \mathbf{Z}^2$ ;
    - $n = -1$  :  $\langle a, b : bab^{-1} = a^{-1} \rangle$  :  
 this group contains a subgroup  $\cong \mathbf{Z}^2$  of index two,  
 but it is not  $\cong \mathbf{Z}^2$ ;
    - 3<sub>b</sub>.  $n \neq 0, \pm 1$  :  $\langle a, b : bab^{-1} = a^n \rangle$  :  
 these groups are 2 step-solvable and of exponential  
 growth (pairwise non-isomorphic).

Also Tits' alternative does hold for one-relator groups: any one-relator group either contains a free subgroup of rank two or is solvable (and from the above list).

But in the Karrass-Solitar paper no algorithm is given answering the question whether, given a one-relator presentation, the corresponding group is solvable or not. In Section 1 we present a simple algorithm and, as a consequence, we re-obtain the above list of all amenable one-relator groups.

In the second part of the paper we investigate the uniformly exponential growth for one-relator groups of exponential growth.

Recall that if  $G$  is a group with a finite generating system  $A$ ,

$$|g|_A = \min \{n : g = a_1 a_2 \cdots a_n, a_i \in A\}$$

is the *length* of an element  $g \in G$  with respect to  $A$  and  $\gamma_A^G(n) = |\{g \in G : |g|_A \leq n\}|$  is the *growth function* of  $G$  with respect to the generating system  $A$ . The limit

$$\lambda_A(G) = \lim_{n \rightarrow \infty} \sqrt[n]{\gamma_A^G(n)}$$

exists and  $\lambda_A(G) \geq 1$ . The group  $G$  is said to have *exponential growth* (respectively *sub-exponential growth*) if  $\lambda_A(G) > 1$  (resp.  $\lambda_A(G) = 1$ ) for some (and therefore for any other) finite system of generators  $A$ .

Denoting now by

$$\lambda_*(G) = \inf_A \lambda_A(G)$$

the *minimal growth rate* of  $G$ , where the infimum is taken over all finite generating systems, the group  $G$  has *uniform exponential growth* if  $\lambda_*(G) > 1$ . This last concept is due to Avez [A] where the number

$$h(G) = \log(\lambda_*(G))$$

is called the *entropy* of the group  $G$  and it is discussed in [GrLP], [SW] and in the survey paper [GH].

The simplest example of a group with uniformly exponential growth is the free group  $\mathbf{F}_m$  of finite rank  $m \geq 2$  for which the minimal growth rate is  $\lambda_*(\mathbf{F}_m) = 2m - 1$ , see for instance [GH].

It is not known whether a group of exponential growth has necessarily uniformly exponential growth or not. We formulate the following:

0.1. CONJECTURE. *All one-relator groups of exponential growth have uniformly exponential growth.*

Conjecture 0.1 is true for one-relator groups of rank  $m \geq 3$  and for one-relator groups with torsion, therefore we focus our attention on two-generated one-relator groups and give sufficient conditions for such groups to have uniformly exponential growth. We present a new method for estimating the minimal growth rate of a finitely generated group using growth functions of the corresponding graded Lie algebra and apply it to one-relator groups.

## 1. AN ALGORITHM FOR CHECKING AMENABILITY

Let  $G$  be a one-relator group with presentation  $(*)$ ; the number  $m$  of the generators of  $G$  in the presentation is called the rank of the presentation. Until Section 4 we shall assume that  $R$  is cyclically reduced and non trivial.

The next observation is well known. We shall include the proof stressing the algorithmic aspect of the statement.

1.1. LEMMA. *Let  $G = \langle a, b, \dots : R(a, b, \dots) \rangle$  be a one-relator group with at least two generators. Then  $G$  has a presentation  $\langle t, \dots : R'(t, \dots) \rangle$  with  $\sigma_t(R') = 0$ , where  $\sigma_t(R')$  denotes the sum of the exponents of  $t$  in the word  $R'$ . This second presentation can in fact be produced, starting from the original one, in an algorithmical way.*