1. About finiteness

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1. About finiteness

Segre's heuristic starting point was that a complete linear system of arithmetic genus g on a smooth quartic is also of dimension g. Now, demanding that the system have a member with g ordinary double points should represent g independent conditions. Hence, for every positive integer h, there should exist a finite number of rational irreducible curves in the complete linear system cut out by the family of all surfaces of degree h.

This argument is unsatisfactory as it stands, because there are always infinitely many *reducible* curves with the right number of nodes (unless $h = 1$). For instance, for $h = 2$ the curve cut by the union of a tritangent plane and any one of the infinitely many *bitangent* planes also has nine double points.

In fact, Segre did try to justify his claim, but in a totally unconvincing way. There is no discussion of the irreducibility of the relevant incidence correspondences. What is more, the argument depends on computing self-intersection numbers on a rather unspecified system of quartic surfaces, all of which might in particular be singular. However, this part of Segre's argument can be replaced by the following lemma. We always work over an algebraically closed ground field k of characteristic 0.

LEMMA 1.1. *No K3 surface carries any one-dimensional (non-constant) algebraic system of irreducible rational curves, whether smooth or singular.*

Proof. A smooth rational curve Γ on a K3 surface has arithmetic genus zero, and hence $(\Gamma)^2 = -2$. Therefore Γ is not even numerically equivalent to any other irreducible curve. This yields the assertion for smooth curves.

As for the singular case, a proof is sketched in [G-G] (Lemma 4.2), but we supply more details. Let B be a parameter curve for an algebraic system of curves on a K3 surface X, whose generic member is irreducible and rational. Without loss of generality, B is irreducible and even smooth, since we are free to replace it by its normalization. Let $\mathcal{J} \subset X \times B$ be the subvariety of codimension 1 corresponding to the algebraic system. Then, by [Sh] (Chap. 1, §6, Thm. 8), $\mathcal{J}$ is irreducible. We denote by $p\colon \mathcal{J} \to X$ the first projection and observe that p is dominant, unless the family is constant.

Let η be a generic point of B over the ground field k, so that $k(\eta) = k(B)$. By assumption, the fibre Γ_η of $\mathcal{J}$ above η is rational over the algebraic closure $\overline{k(\eta)}$ of $k(\eta)$. So, by a result which goes back to Hilbert and Hurwitz, Γ_η is birationally equivalent over $k(\eta)$ to a smooth conic. Now, k is algebraically closed; so, if t is a variable then $k(t)$ is a C_1 field (*cf.* [La], Thm. 6).

As the function field of some curve, $k(\eta)$ is an algebraic extension of $k(t)$; hence it is also C_1 ([La], pp. 376–377). So, every conic defined over $k(\eta)$ has points defined over this field and is birationally equivalent to $\mathbf{P}^1_{k(\eta)}$. This shows that $k(\eta)(\Gamma_\eta)$ is isomorphic to $k(\eta)(t)$. Therefore we have the following k-isomorphisms:

$$k(\mathcal{J}) \approx k(\eta)(\Gamma_\eta) \approx k(\eta)(t) = k(B \times \mathbf{P}^1).$$

Hence there is a birational equivalence $\varphi \colon B \times \mathbf{P}^1 \dashrightarrow \mathcal{J}$. Consider the composite rational map $q = p \circ \varphi \colon B \times \mathbf{P}^1 \dashrightarrow X$. Since q is dominant, and X projective, we know (*cf.* [Sh], Chap. 3, §5, Thm. 2) that q^* embeds the regular differentials (of any rank) on X into those on $B \times \mathbf{P}^1$.

Since X is a K3 surface, we note that ω_X is trivial, and hence $h^0(\omega_X) = 1$. On applying q^* we see that $h^0(B \times \mathbf{P}^1, \omega_{B\times\mathbf{P}^1}) \neq 0$. But this is impossible. Indeed, if we denote by p_1 and p_2 the projections from $B \times \mathbf{P}^1$ to B and $\mathbf{P}^1$ respectively, we have:

$$\omega_{B\times\mathbf{P}^1} = p_1^*\omega_B \otimes p_2^*\omega_{\mathbf{P}^1}.$$

On the other hand, $H^0(\mathbf{P}^1, \omega_{\mathbf{P}^1}) = H^0\big(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(-2)\big) = 0$, and for quasi-coherent sheaves the global section functor commutes with tensor products; a contradiction. □

REMARK. Lemma 1.1 does not imply that a given K3 surface cannot contain infinitely many smooth rational curves; see [SwD], §5, for an example.

2. About existence

Finiteness statements are useless if they are not accompanied by some form of existence assertion. After all, zero is also a finite number! In the present section we show the existence of irreducible rational curves, of degree 8 or 12, at least on some smooth quartic surfaces. For degree 8 there is a very elementary proof, and we give it first. Then we shall proceed to the case of degree 12, which requires some more elaborate machinery.

As mentioned above, on a quartic surface it is easy to find some reducible curves of degree 8 with nine double points by considering unions of two plane sections. Such curves are even infinite in number, but they do not lie on any smooth quadric.[4]) That is why we start with a very explicit construction on

[4]) By the way, this may be one reason for working with the Chow variety rather than with a Hilbert scheme. These degenerate cases have the same arithmetic genus, but they do not lie in $\mathcal{R}_8$.