§2. Proof of Ingham's Theorem

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **43 (1997)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **26.09.2024**

§2. Proof of Ingham's Theorem

Suppose that $g(1) = 0$. We show that this assumption leads to a contradiction. We consider the function

$$F(s) = \zeta(s)^2 \, g(s) \, g^*(s) \qquad (\operatorname{Re} s > 1),$$

where $g^*(s) = g(s, \overline{\epsilon})$ and $\overline{\epsilon}$ is the arithmetic function which is the complex conjugate of ϵ. Clearly $g^*(1) = \overline{g(1)} = 0$. By hypothesis g is regular along the stretch $\left[\frac{1}{2}, 1\right]$ of the real axis and so therefore is g^*, since $g^*(s) = \overline{g(\overline{s})}$. Hence F is regular on $\left[\frac{1}{2}, 1\right]$, since the double pole of ζ^2 at $s = 1$ is canceled by the zeros of g and g^* there.

Using the identity

$$(1 - z)^{-1} = \exp\left(\sum_{k=1}^{\infty} z^k/k\right) \qquad (|z| < 1),$$

we obtain (for $\operatorname{Re} s > 1$)

$$\begin{aligned} F(s) &= \prod_p \left(1 - \frac{1}{p^s}\right)^{-2} \left(1 - \frac{\epsilon(p)}{p^s}\right)^{-1} \left(1 - \frac{\overline{\epsilon}(p)}{p^s}\right)^{-1} \\ &= \prod_p \exp\left(\sum_{k=1}^{\infty} \frac{2 + \epsilon(p)^k + \overline{\epsilon}(p)^k}{kp^{ks}}\right) \\ &= \prod_p \left\{1 + \left(\sum_{k=1}^{\infty} \frac{2 + \epsilon(p)^k + \overline{\epsilon}(p)^k}{kp^{ks}}\right) + \frac{1}{2}\left(\sum_{k=1}^{\infty} \frac{2 + \epsilon(p)^k + \overline{\epsilon}(p)^k}{kp^{ks}}\right)^2 + \cdots\right\}. \end{aligned}$$

Thus F has a Dirichlet series expansion

$$F(s) = \sum_{n=1}^{\infty} a(n) \, n^{-s} \qquad (\operatorname{Re} s > 1).$$

Furthermore, since

$$2 + \epsilon(p)^k + \overline{\epsilon}(p)^k = 2 + 2\operatorname{Re}\{\epsilon(p)^k\} \geq 0,$$

we have $a(n) \geq 0$ for all n.

At this point we deviate from the approach used in [4] and [5] by noting that $a(p^2) \geq 1$ for each prime p. For, since

$$F(s) = \prod_p \left(1 + \frac{2}{p^s} + \frac{3}{p^{2s}} + \cdots\right)\left(1 + \frac{\epsilon(p)}{p^s} + \frac{\epsilon(p)^2}{p^{2s}} + \cdots\right)\left(1 + \frac{\overline{\epsilon}(p)}{p^s} + \frac{\overline{\epsilon}(p)^2}{p^{2s}} + \cdots\right),$$

we find that

$$\begin{aligned} a(p^2) &= 3 + 2\epsilon(p) + 2\bar{\epsilon}(p) + \epsilon(p)^2 + \epsilon(p)\,\bar{\epsilon}(p) + \bar{\epsilon}(p)^2 \\ &= 2 - \epsilon(p)\,\bar{\epsilon}(p) + \{1 + \epsilon(p) + \bar{\epsilon}(p)\}^2 \\ &\geq 2 - |\epsilon(p)|^2 \geq 1\,. \end{aligned}$$

Thus

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^{1/2}} \geq \sum_{p} \frac{a(p^2)}{p} \geq \sum_{p} \frac{1}{p}\,.$$

In view of the divergence of $\sum p^{-1}$, it follows that $\sum a(n)\,n^{-1/2}$ diverges.

On the other hand, applying Landau's lemma with $c_n = a(n)$, $\beta = \frac{1}{2}$, $\gamma = 1$, we find that $\sum a(n)\,n^{-1/2}$ converges. This contradiction shows that the assumption $g(1) = 0$ is untenable and so the proof is complete.

REFERENCES

[1] INGHAM, A. E. Note on Riemann's ζ-function and Dirichlet's L-functions. *J. London Math. Soc. 5* (1930), 107–112.

[2] LANDAU, Edmund. Über einen Satz von Tschebyschef. *Math. Ann. 61* (1905), 527–550, or *Collected Works vol. 2* (Thales Verlag, 1986), 206–229.

[3] —— Über das Nichtverschwinden einer Dirichletschen Reihe. *Sitzungsberichte der Preussischen Akad. Wiss. Berlin* (1906), 314–320, or *Collected Works vol. 2* (Thales Verlag, 1986), 230–236.

[4] NARASIMHAN, Raghavan. Une remarque sur $\zeta(1 + it)$. *L'Enseignement Math. (2) 14* (1969), 189–191.

[5] NEWMAN, D. J. A "natural" proof of the non-vanishing of L-series. *Contemp. Math. 143* (1993), 495–498.

[6] SHAPIRO, George. On the non-vanishing at $s = 1$ of certain Dirichlet series. *Amer. J. Math. 71* (1949), 621–626.

(Reçu le 16 avril 1997)

Paul T. Bateman

Department of Mathematics
University of Illinois
Urbana, IL 61801
U. S. A.
e-mail: bateman@math.uiuc.edu