5. Conservative transverse line fields

REMARK. The following result is also known (see the literature cited): if a convex closed curve intersects a curve, homothetic to J, at $2n$ points then it has at least $2n$ Minkowski vertices.

5. CONSERVATIVE TRANSVERSE LINE FIELDS

In this section we discuss the following problem: given a smooth strictly convex closed plane curve γ and a smooth transverse line field l along it, when does a parameterization $\gamma(t)$ exist such that the line $l(t)$ at point $\gamma(t)$ is generated by the acceleration vector $\gamma''(t)$ for all t?

DEFINITION. A transverse line field along a closed plane curve, generated by the acceleration vectors for some parameterization of the curve, is called *conservative*.

Clearly, not every line field is conservative: consider, for example, a field of lines that everywhere make an acute angle with the curve. Theorem 0.1 provides a necessary condition: the envelope of the lines from a conservative line field has at least 4 cusps. Lemma 3.2 gives another one: there exist at least 2 tangent lines to this envelope through every point in the plane.

We start with the following situation. Let M^3 be a contact manifold and let $\widetilde{\gamma} \subset M$ be a closed smooth Legendrian curve. Recall that the characteristic line field η of a contact form λ is the field $Ker\ d\lambda$. Assume that the contact distribution along $\widetilde{\gamma}$ is coorientable; then it can be determined by a contact form. Let η be a line field along $\widetilde{\gamma}$, transverse to the contact distribution.

QUESTION. When does a contact form exist in a vicinity of $\widetilde{\gamma}$ for which η is the characteristic field?

When this is the case we call the field η *characteristic*.

Let λ be some contact form near $\widetilde{\gamma}$ and let v be a vector field along $\widetilde{\gamma}$ that generates the line field η. Consider the 1-form $(i_v\ d\lambda)/\lambda(v)$ and set

$$\beta(\widetilde{\gamma}, \eta) = \int_{\widetilde{\gamma}} \frac{i_v\ d\lambda}{\lambda(v)}\,.$$

THEOREM 5.1. *The number $\beta(\widetilde{\gamma}, \eta)$ does not depend on the choice of the contact form λ nor the vector field v. This number vanishes if and only if the field η is characteristic.*

Proof. Clearly, $(i_v\ d\lambda)/\lambda(v)$ does not change if v is multiplied by a nonvanishing function. Let $\lambda_1 = f\lambda$ with $f \neq 0$ be another contact form. Then $d\lambda_1 = df \wedge \lambda + f d\lambda$. One has

$$\begin{aligned}\int_{\tilde{\gamma}} \frac{i_v\ d\lambda_1}{\lambda_1(v)} &= \int_{\tilde{\gamma}} \frac{f\ i_v\ d\lambda + df(v)\ \lambda - \lambda(v)\ df}{f\ \lambda(v)} \\ &= \int_{\tilde{\gamma}} \frac{i_v\ d\lambda}{\lambda(v)} + \int_{\tilde{\gamma}} \frac{df(v)}{f\ \lambda(v)} \lambda - \int_{\tilde{\gamma}} \frac{df}{f} .\end{aligned}$$

The second integral on the right hand side vanishes because $\tilde{\gamma}$ is a Legendrian curve, tangent to the kernel of $df(v)\lambda/f\lambda(v)$, and so does the third because df/f is an exact 1-form. Thus $\beta(\tilde{\gamma}, \eta)$ does not depend on the choices involved.

If η is characteristic for a contact form λ then $i_v\ d\lambda = 0$, so $\beta(\tilde{\gamma}, \eta) = 0$. Conversely, let $\beta(\tilde{\gamma}, \eta) = 0$. A neighbourhood of $\tilde{\gamma}$ in M is contactomorphic to a neighbourhood of the zero section in the space of 1-jets $J^1 S^1$ (see [A 3]). That is, there exist coordinates (x, y, z), $x \in S^1$, $y, z \in \mathbf{R}^1$ in which the contact structure is given by the 1-form $\lambda_0 = dz - y dx$, and $\tilde{\gamma}$ is the curve $y = z = 0$. Since η is transverse to the contact structure one may assume it to be generated by the vector field

$$v = a(x)\ \partial/\partial x + b(x)\ \partial/\partial y + \partial/\partial z ,$$

where $a(x)$ and $b(x)$ are functions on the circle.

Then

$$\beta(\tilde{\gamma}, \eta) = \int_{\tilde{\gamma}} \frac{i_v\ d\lambda_0}{\lambda_0(v)} = -\int b(x)\ dx .$$

If $\beta(\tilde{\gamma}, \eta)$ vanishes then there exists a function $g(x)$ such that $b(x) = g'(x)$. Next, a direct computation shows that the characteristic line field of the contact form $e^{f(x,y,z)}\ \lambda_0$ is generated by the vector field

$$f_y\ \partial/\partial x\ - (f_x + y f_z)\ \partial/\partial y + (1 + y f_y)\ \partial/\partial z ,$$

which equals, along $\tilde{\gamma}$,

$$u = f_y\ \partial/\partial x - f_x\ \partial/\partial y + \partial/\partial z .$$

Therefore, setting $f(x, y, z) = a(x) y - g(x)$, one has: $v = u$, and the field η is characteristic.

Thus the characteristic line fields constitute a codimension 1 subspace in the (infinite dimensional) space of line fields along $\tilde{\gamma}$, transverse to the contact structure.

Return to the situation at the beginning of the section. Let γ be a smooth strictly convex closed curve, cooriented inwards, and let l be a smooth transverse line field along γ. As before, $\widetilde{\gamma}$ is the Legendrian curve in the space of cooriented contact elements $ST^*\mathbf{R}^2$, corresponding to γ. For every point $x \in \gamma$ consider the family of cooriented contact elements along the line $l(x)$, parallel to the contact element of γ at x. This gives a line field η along $\widetilde{\gamma}$, a lift of the field l. The field η is transverse to the contact structure.

Choose a parameterization $\gamma(t)$, $0 \le t \le T$, and a vector field $u(t)$ along γ that generates the line field $l(t)$.

LEMMA 5.2. *One has:*

$$\beta(\widetilde{\gamma}, \eta) = \int_0^T \frac{[\gamma''(t), u(t)]}{[\gamma'(t), u(t)]}\, dt .$$

Proof. Let v be the lift of u to $ST^*\mathbf{R}^2$ that generates the field η. In Theorem 2.1 a Hamiltonian function H in $ST^*\mathbf{R}^2$ is constructed, associated with the parameterization $\gamma(t)$ (one does not need the assumption $[\gamma''(t), \gamma'''(t)] \neq 0$ here). The space $ST^*\mathbf{R}^2$ is identified with $\mathbf{R}^2 \times S$, where the star-shaped curve $S \subset (\mathbf{R}^2)^*$, the level curve of H, consists of the covectors $[\gamma'(t), \ \ ]$. The corresponding contact form λ is the restriction of the Liouville form pdq to $\mathbf{R}^2 \times S$. The curve $\widetilde{\gamma}$ is given by the formula:

$$\widetilde{\gamma}(t) = (\gamma(t), \ [\gamma'(t), \ \ ]) .$$

It follows that $\lambda(v(t)) = [\gamma'(t), u(t)]$. Likewise,

$$(i_{v(t)} d\lambda)(\widetilde{\gamma}'(t)) = (i_{v(t)} dp \wedge dq)(\widetilde{\gamma}'(t)) = [\gamma''(t), u(t)] .$$

Therefore

$$\int_{\widetilde{\gamma}} \frac{i_v \, d\lambda}{\lambda(v)} = \int_0^T \frac{[\gamma''(t), u(t)]}{[\gamma'(t), u(t)]}\, dt .$$

The lemma is proved.

In particular, the value of the integral

$$\int_0^T \frac{[\gamma''(t), u(t)]}{[\gamma'(t), u(t)]}\, dt$$

does not depend on the parameterization $\gamma(t)$ nor on the choice of the vector field $u(t)$. Denote this integral by $\alpha(\gamma, l)$.

LEMMA 5.3. *The line field l along γ is conservative if and only if the line field η along $\widetilde{\gamma}$ is characteristic.*

Proof. If l is generated by the vectors $\gamma''(t)$ then η consists of the characteristic directions of the contact form in $ST^*\mathbf{R}^2$, associated with the parameterization $\gamma(t)$ in Theorem 2.1 (cf. the proof of the preceding lemma).

Conversely, a contact form λ along $\widetilde{\gamma}$, whose characteristics are the lines η, is a field of covectors p along γ which vanish on the tangent lines to γ at the respective points. Define the parameterization $\gamma(t)$ by the condition: $[\gamma'(t),\quad] = p\ (\gamma(t))$ for all t. Then the contact form in $ST^*\mathbf{R}^2$, associated with this parameterization according to Theorem 2.1, coincides with λ along $\widetilde{\gamma}$. Therefore the lines $l(t)$ are generated by the vectors $\gamma''(t)$.

Combining Theorem 5.1, Lemma 5.2 and 5.3, one arrives at the following result (discovered in [T 2] and proved therein by a direct computation).

THEOREM 5.4. *A transverse line field l along a smooth strictly convex closed plane curve γ is conservative if and only if $\alpha(\gamma, l) = 0$.*

Thus conservative line fields constitute a codimension one subspace in the space of transverse line fields along a closed curve.

EXAMPLE. L. Guieu and V. Ovsienko studied the following situation in [G-O]. Given a smooth convex closed plane curve consider the field of lines connecting each point of the curve with a focus of its osculating conic at this point (see Example 2 in Section 3). This line field is conservative, and its envelope, called the gravitational caustic in [G-O], has at least 6 cusps.

Consider a curve γ with a transverse line field l. A (partial) diffeomorphism of the plane F takes γ to a new curve $F(\gamma)$ with the transverse line field $dF(l)$. The field $dF(l)$ does not have to be conservative even if l is.

EXAMPLE. Let γ be the unit circle, l consists of its normals, and F is given near γ in polar coordinates by the formula: $(\alpha, r) \to (\alpha + r, r)$. Then $F(\gamma) = \gamma$, and the lines $dF(l)$ make a constant acute angle with the circle.

However the following result holds (to answer a question by V. Arnold).

THEOREM 5.5. *Every projective transformation of the plane takes the conservative line fields to the conservative ones.*

Proof. Consider $\mathbf{R}^2$ as the plane $\{z = 1\}$ in Euclidean 3-space, and let

$$\pi : (x, y, z) \to (x/z, y/z)$$

be the projection of the half-space $\mathbf{R}^3_+ = \{z > 0\}$ on $\mathbf{R}^2$. Consider a parametrized curve $\Gamma(t) \subset \mathbf{R}^3_+$, and let $\gamma(t) = \pi(\Gamma(t))$.

Claim: the field $(d\pi)(\Gamma''(t))$ is conservative along the curve $\gamma(t)$.

Indeed, a direct computation (which is left to the reader) shows that

$$(d\pi)(\Gamma''(t)) = \gamma''(t) + 2\,\frac{z'(t)}{z(t)}\,\gamma'(t)\cdot$$

Therefore

$$\alpha(\gamma,\ (d\pi)(\Gamma''(t))) = -\int 2\,\frac{z'(t)}{z(t)}\,dt = -2\int d\ \log\ z(t) = 0\,.$$

The claim follows from Theorem 5.4.

Let A be a linear transformation of space. Then $F = \pi A : \mathbf{R}^2 \to \mathbf{R}^2$ is a projective transformation, and all projective transformations are obtained this way. Consider a curve $\gamma(t) \subset \mathbf{R}^2$, and let $l(t)$ be generated by the acceleration vectors $\gamma''(t)$. Let $\Gamma(t) = A(\gamma(t))$; assume, without loss of generality, that $\Gamma(t) \subset \mathbf{R}^3_+$ One has: $\Gamma''(t) = A(\gamma''(t))$, and it follows from the above claim that the field $(d\pi)(\Gamma''(t))$ is conservative along the curve $\pi(\Gamma(t))$. Thus the line field $dF(l)$ is conservative along the curve $F(\gamma)$.

REMARK. Theorem 5.5 shows that the notion of the conservative line fields along closed curves is a projective, and not an affine, one. Thus one hopes that the theory of this paper can be extended to spherical curves in the spirit of [A 5].

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ADDED IN PROOF. A higher dimensional analog of conservative transverse line fields is studied in the author's paper "Exact transverse line fields and projective billiards in a ball", to appear in "Geometric and Functional Analysis".

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